Fast implemented block adaptive preconditioning scheme for the forward problem in Electrical Impedance Tomography

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***Abstract*— The forward problem in Electrical Impedance Tomography (EIT) is the one of recovering the coefficients of a large sparse and positive definite matrix, obtained after discretization of a partial differential equation. The problem is said to be ill-conditioned which, amongst others, reflects a very slow convergence speed for the solution. Preconditioning the forward problem using incomplete Cholesky factorisation is the natural approach to solving the problem efficiently, having as a main drawback that its calculation is not always stable. Our work proposes a block adaptive preconditioning method which takes the advantage of the standard structure of the coefficients matrix to deliver a better conditioned system, yielding to a stable and fast computation of the preconditioning matrix for an efficient solution. Numerical results demonstrate the effectiveness of our proposed approach.**

I. Introduction

Electrical Impedance Tomography (EIT) is a relatively new and promising imaging modality, already used in industrial applications that can be extensively used for monitoring in various medical applications. In a similar way to many other physical phenomena that are described by a partial differential equation, a set of equations is obtained as long as the domain is discretized. By transforming the equations into a standard matrix form, $Ax = b$, a large, sparse and (mainly) positive definite matrix is derived, called the coefficients or sensitivity matrix, often characterized by a very high condition number [1]. Hence, a form of preconditioning is needed to transform the original system of equations, in order to obtain a fast convergence and an efficient solution. However, as the scale of the system becomes larger (i.e., finer discretization) the convergence speed becomes a very crucial factor for many applications.

In this paper, we employ a preconditioning scheme initially introduced in [2], [3] to derive a better-conditioned system for the forward problem in EIT. In the next section, a brief introduction to EIT is given, with emphasis on the structure of the coefficients matrix. Next, in section three, standard preconditioning techniques are introduced with respect to incomplete Cholesky factorisation, while section four describes the proposed block adaptive preconditioner. Numerical simulations in section five are based on a 3D model consisting of 3262 first-order tetrahedral elements, originally derived for the application of monitoring dysphagia, a swallowing abnormality.

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II. Problem Formulation

In EIT, the internal conductivity structure of a body is reconstructed when electrodes are placed along the periphery of an object and current patterns are applied to some of them, while the rest are collecting the corresponding measurements. The imaging capabilities of EIT are based on the knowledge of a finite element model and, in general, in the a-priori knowledge about the solution. Thus, for a conductive volume of fixed boundaries and an initial conductivity distribution, the forward problem requires the calculation of the potential distribution inside the volume from measurements on the boundaries. The problem is mainly solved numerically than analytically and in particular with the use of Finite Elements Methods (FEM). The mathematical model that is used to solve the problem is based on a simplified version of the Maxwell's equations engaged with the complete electrode model [4]. Using finite elements and according to the complete electrode model, the coefficients matrix can be assembled and structured as follows [5]

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{1}$$

which eventually describes a linear system of equations of the form $Ax = b$, where ,in general, the compartment A_{11}, refers to the problem formulation with natural boundary conditions, whereas compartments A_{12}, A_{21} and A_{22} superimpose and form the boundary conditions under the complete electrode model conditions. A sparse plot of the coefficient matrix is shown in Fig. 1.

III. Preconditioning

In general, preconditioning is a technique for improving the condition number of a matrix. Assume that M is a symmetric positive definite matrix that approximates the coefficient matrix A however is easier to invert. In the general case, one can solve a problem of the form $Ax = b$ indirectly by solving

$$M^{-1}Ax = M^{-1}b \tag{2}$$

As, A is not well-conditioned, the goal is to achieve a condition number κ such that $\kappa(M^{-1}A) \ll \kappa(A)$ or in other words, the eigenvalues of $(M^{-1}A)$ to be better clustered in order to achieve a faster convergence rate for the forward problem, than those of A. Then, solve Eq. (2) iteratively rather than the original equation.

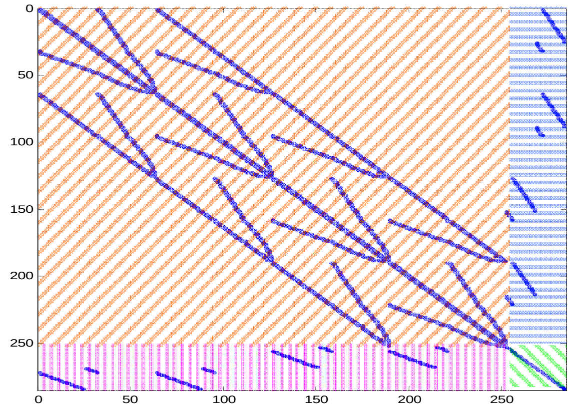

Fig.1 Structure of Coefficient matrix where the orange, blue, pink and green shaded regions corresponds to A_{11}, A_{12}, A_{21} and A_{22} compartments respectively.

In the general case, the main drawback when preconditioning is that $(M^{-1}A)$ is neither generally symmetric nor positive definite even when M and A are. However, this difficulty can be overcome by considering that every positive definite symmetric matrix M can be (not uniquely) decomposed, using, for instance, complete or incomplete Cholesky factorization, as $EE^T = M$ and hence transforming the original forward problem into

$$E^{-1}AE^{-T}\tilde{x} = E^{-1}b \tag{3}$$

where $\tilde{x} = E^T x$.

Eq. (3) is initially solved for $\tilde{x}$ and then for x. As $E^{-1}AE^{-T}$ is symmetric and positive definite, it can be efficiently solved using Conjugate Gradient (CG) [6].

A good measure of the effectiveness of a preconditioner is determined by the condition number of $(M^{-1}A)$ and hence the initial problem is restated as finding a preconditioning scheme which better approximates the system matrix A in order to improve convergence. The ideal preconditioner is $M = A$, as $(M^{-1}A)$ has a condition number of one, while the simplest preconditioner is a diagonal matrix, whose diagonal entries are identical to those of A, as it is quite trivial to invert. However, it is often an average performance preconditioner.

On the other hand, the most practical and popular selection is the incomplete Cholesky preconditioner [7], defined for the coefficients matrix as $A = LL^T$, where L is a lower triangular matrix for Cholesky factorization and $A = \tilde{L}\tilde{L}^T$ is the incomplete Cholesky factorisation, where little or no fill is required. Unfortunately, incomplete Cholesky factorization is not always stable and may not deliver the fastest approximation for the system matrix, in terms of the implementation efficiency for large scale applications. If, for instance, the factorization fails due to pivot breakdown, a global diagonal shift should be applied to A prior to reattempting it. Further, $\tilde{L}$ may be restricted to have the same pattern of non-zero elements as A, where the other elements of L are thrown away.

IV. Adaptive Preconditioning

An attractive feature for selecting an adaptive preconditioner is the idea of subspace invariance in order to deflate the eigenvalues of the system matrix associated with this subspace, resulting in a more compact eigenvalue distribution and thus a better convergence rate. The drawback when using this approach is that in real applications, the construction of such subspaces can be rather costly. Let the following block triangular preconditioner be defined as [3]

$$P_r = \begin{pmatrix} A_{11} & A_{12} \\ 0 & \tilde{S} \end{pmatrix} \tag{4}$$

where $\tilde{S}$ is an approximattion to the Shur complement $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ [8]. The proposed preconditioner can be easily computed as blocks A_{11}, A_{12} are already pre-calculated to form the system matrix. We assume that the Shur complement can be easily calculated if an efficient solver for the inversion of A_{11}^{-1} is employed and hence calculation of P_r reduces to a trivial task.

V. Simulation Overview

One of the challenging applications in EIT apart from cancer tumour and brain activity detection is monitoring of Dysphagia, a swallowing abnormality usually followed after stroke. Thus, consider a simplified mesh model of isotropic conductivity, which demonstrates the geometry of the given application, i.e., the human neck. The mesh is generated with the use of NETGEN, an open source free element discretization software under the Lesser General Public License (LGPL), while the calculations of the coefficients matrix as well as the reconstructed images were made with the use of EIDORS software [9].

Current patterns are injected to the body and a set of measurements is then obtained, given the different electrode configurations, yielding to a subset of linearly independent measurements. The calculated coefficients matrix has a condition number of $2.4732 \cdot 10^5$. Table 1 shows the effect of preconditioning on the condition number, which is reduced to $1.5489 \cdot 10^5$, when incomplete Cholesky preconditioning is used.

Moreover, the proposed adaptive preconditiong scheme drastically reduces the condition number to 1.3691 and delivers a very compact cluster for the singular values, as shown in semi-logarithimic scale in Fig. 3.

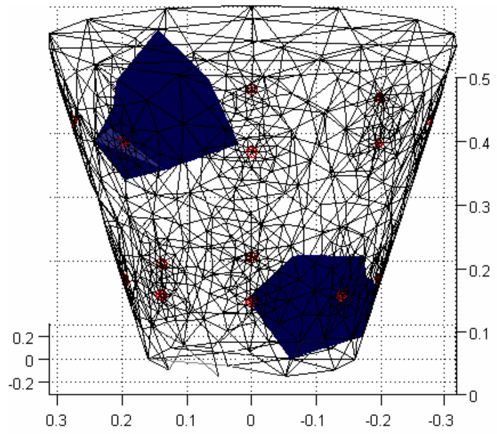

Fig. 2 A conical mesh consisting of 32562 first order tetrahedral elements

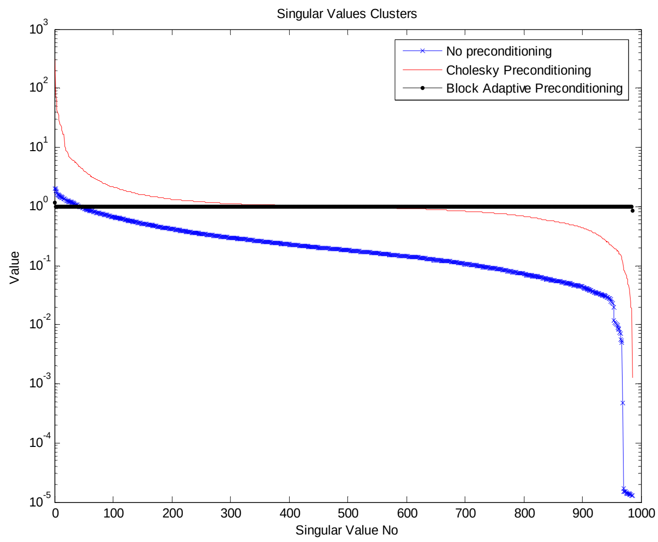

Fig. 3 Singular values plot for different preconditioning schemes

As shown in Fig.3, block preconditioning results in a much better conditioned matrix and hence, in a faster iterative solution. The simulated inhomogeneities obtained when the proposed preconditioning scheme is applied to the forward problem are shown in the first and third columns in Fig. 4, whereas the reconstructed ones, using the default EIDORS regularized solution based on the face smoothing solver, in the second and fourth columns, respectively.

VI. Conclusions

The forward problem in EIT is hindered by the high condition number of the coefficients matrix, leading to a slow convergence rate which, in particular, in large scale applications becomes a crucial factor. A preconditioning scheme was presented along with numerical results to dramatically improve the performance of the forward problem and result in drastically reduced condition number for the coefficient matrix, under the assumption that the conductivity is real. Future work in this direction suggests evaluation of the proposed scheme amongst different solvers to derive an effective approach for the solution of the forward problem in EIT for both real and complex admittivities. With regard to the specific application, this is the first time, to our knowledge, that a fine 3D model towards monitoring of Dysphagia is presented.

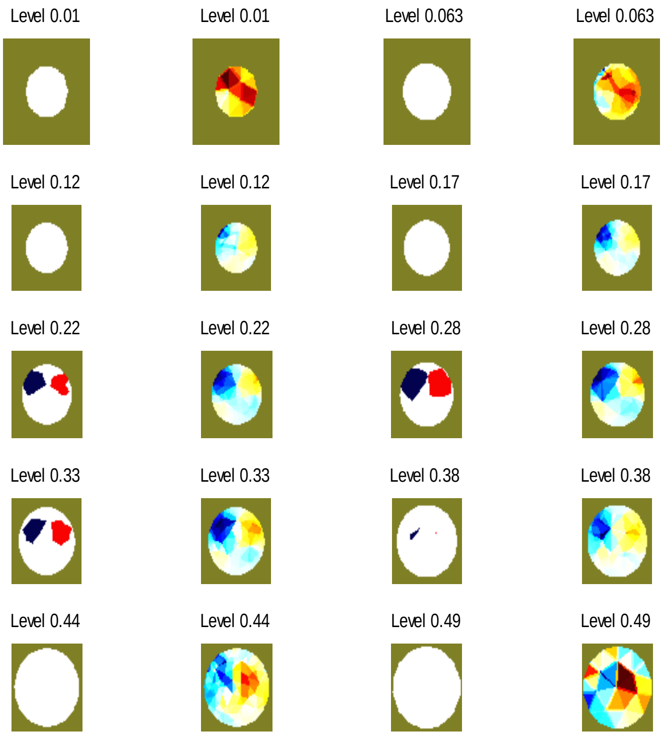

Fig. 4 Simulated and reconstructed images in first, third and second, fourth, columns, respectively.

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