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Solving of L0 norm constrained EEG inverse problem

Peng Xu, Xu Lei, Xiao Hu *Member, IEEE*, Dezhong Yao *Member, IEEE*

***Abstract*—** l_0 **norm is an effective constraint used to solve EEG inverse problem for a sparse solution. However, due to the discontinuous and un-differentiable properties, it is an open issue to solve the** l_0 **norm constrained problem, which is usually instead solved by using some alternative functions like** l_1 **norm to approximate** l_0 **norm. In this paper, a continuous and differentiable function having the same form as the transfer function of Butterworth low-pass filter is introduced to approximate** l_0 **norm constraint involved in EEG inverse problem. The new approximation based approach was compared with** l_1 **norm and LORETA solutions on a realistic head model using simulated sources. The preliminary results show that this alternative approximation to** l_0 **norm is promising for the estimation of EEG sources with sparse distribution.**

I. Introduction

Current EEG inverse problem usually needs to solve the following underdetermined system [1], [2],

$$Y = AX + \varepsilon \tag{1}$$

where *Y* is recording with size of $M \times 1$, *M* is the number of sensors. *A* is the lead field matrix of $M \times N$ with *N* being the dimension size of possible solution space. *X* is the solution vector to be estimated. EEG inverse problem is usually underdetermined with *M* much smaller than *N*, thus the problem lacks a unique solution because there are an infinite number of possible solution configurations that could explain the measured recordings *Y* [1]-[3]. How to obtain a feasible solution consistent with the actual problem is still the focus in EEG or EMG inverse problem. To solve this underdetermined problem for a solution consistent with the actuality, some constraints are usually imposed. Those constraints such as the minimum norm least square solution (MNS) and the neighbor information of solution space like the Laplacian operator used in low resolution electromagnetic tomography (LORETA) are currently adopted. Sparse constraint is a possible alternative choice. The sparse activation has been observed from some physiological signals. For example, the main neural electric activities are usually sparsely localized, thus a sparsely localized solution may explain the scalp recordings in a more reasonable sense [1], [3], [4], [13]-[15].

The original and most effective metric to measure the sparsity of a signal is the l_0 norm, which is to calculate the number of the non-null entries in the signal [5]. If the l_0 norm of the solution is taken as a constraint, we have a Lagrange multiplier expression of the inverse problem (1) as,

$$\min_X \| Y - AX \|_2 + \lambda \| X \|_0 \tag{2}$$

where λ is the punishment factor of sparsity. However, formula (2) is not continuously differentiated and many effective gradient based optimization methods can not be directly applied to solve this problem, so in practice, $l_p (p \le 1)$ norm is usually adopted to approximate l_0 norm, among which l_1 norm is the most popular choice , whereas the approximation will decrease the solution sparsity to some degree [1], [5], [6]. To solve the l_0 constrained problem more robustly, we proposed to use a new function to approximate the l_0 norm. This new function is derived from the transfer function of a n-order low-pass Butterworth filter, the cutoff frequency of which will then be adaptively adjusted according to the source distribution estimated during the solving iterations.

The method was introduced in Section II. In Section III, the algorithm was compared with LORETA and l_1 norm solution on a 3-shell realistic head model using the simulated source configurations having different number of sources. Discussions and conclusions concluded this paper in Section IV.

II. Method

A. Laplacian weighted minimum norm solution

The weighted minimum norm solution is the mostly adopted one in the current EEG inverse problem [1], [6]. With weight matrix *W*, the weighted form of EEG inverse problem is,

$$Y = AX + \varepsilon = AWW^{+}X + \varepsilon \tag{3}$$

where q is an auxiliary variable. The weighted minimum-norm solution of the inverse problem is,

$$\hat{X} = W^{-1}A^T[AW^{-1}A^T]^{+}Y \tag{4}$$

Peng Xu, Xu Lei and Dezhong Yao are with Key laboratory for NeuroInformation of Ministry of Education, School of Life Science and Technology, University of Electronic Science and Technology of China, ChengDu, Sichuan, 610054, China. (Correspondence author: Dezhong Yao, email: dyao@uestc.edu.cn, Tel: +86-028-83207204)

Xiao Hu is with Neural Systems and Dynamics Laboratory, Department of Neurosurgery, the David Geffen School of Medicine, University of California, Los Angeles, 90095, USA

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where $[AW^{-1}A^T]^+$ denotes the Moore-Penrose pseudo-inverse of $[AW^{-1}A^T]$. The widely adopted Laplacian weight matrix W in EEG inverse problem has the form as [7], [13], [15],

$$W = BG \text{, with } G = diag(\|a_1\|, \|a_2\|, \ldots\ldots, \|a_N\|) \quad (5)$$

where B denotes the discrete spatial Laplacian operator; $\| a_i \|$ is the *ith* column norm of the lead field matrix A. The solution with such weight strategy is the low resolution electromagnetic tomography (LORETA) and the solution of LORETA is extensive and blurring.

B. l_1 norm solution

The object function for the l_1 norm solution is[1], [6],

$$\min_X \| Y - AX \|_2 + \lambda \| X \|_1 \quad (6)$$

In this paper, Levenberg-Marquardt procedure is used to solve this problem for the l_1 norm solution [8].

C. l_0 norm solution

The low-pass Butterworth filter has form,

$$H(x) = \frac{1}{1 + (x / x_0)^{2n}} \quad (7)$$

where x_0 is the half cutoff frequency, and n defines the filter order. Because minimization is implemented in equation (2), we proposed to use the following function instead of l_1 norm to approximate the l_0 norm term in equation (2),

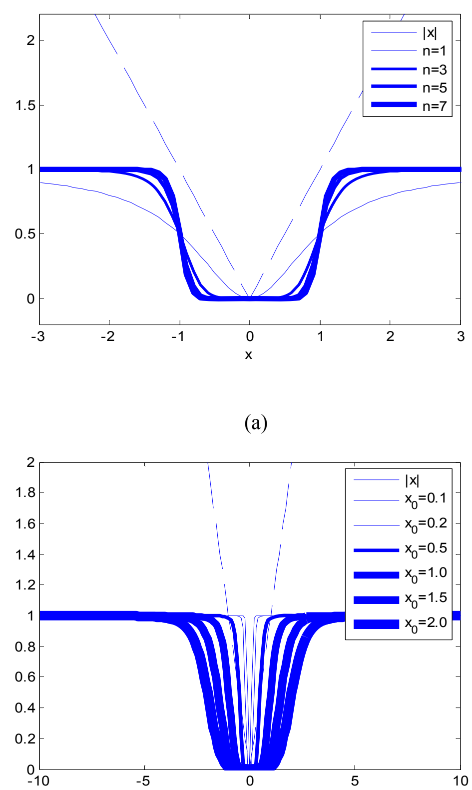
$$G(x) = 1 - \frac{1}{1 + (x / x_0)^{2n}} \quad (8)$$

The curves of l_1 norm and G(x) with different n's are shown in Fig.1(a). Figure 1(b) shows the curves of $G(x)$ with different x_0's.

As shown in Fig.1, the values of $G(x)$ are very close to zero when x varies within a small range closely enclosed the origin of x-axis, and this small range is modulated by the selection of half cutoff frequency x_0, i.e., a small x_0 for a narrow notch and vice verse. When x moves away from origin, $G(x)$ will quickly change toward one. The effect of x_0 can be obviously observed in Fig.1(b), where the approximation to l_0 norm is well conserved with a very narrow notch when x_0 is small. Fig.1(a) shows that the order n will influence the ascending or descending slopes of the notch, and a large order will facilitate a steep slope. The difference between l_1 norm and *G(x)* can be revealed in Fig.1. No upper bound for l_1 norm and the *G(x)* behaves like a threshold function, which will map those x's with small absolute values to around zeros and map those x's having large absolute values to approximate ones. Because the ideal response of l_0 norm is a notch impulse at origin, the values of which are all one except for zero at origin. Herein, *G(x)* is a more meaningful approximation to l_0 norm related term in equation (2). With this approximation, the EEG inverse problem denoted in (2) can be transformed to,

$$\min_X \| Y - AX \|_2 + \lambda G(X) \quad (9)$$

where $G(X) = \| (1 - \frac{1}{1 + (X / x_0)^{2n}}) \|_1$ with $\| \cdot \|_1$ being operator of l_1 norm. Because *G(x)* is continuous and differential in the whole domain, it is very easy to solve this problem with many effective gradient based optimization approaches. In this paper, we used Levenberg-Marquardt procedure to solve this problem [8].



(b)

Fig.1. Approximation of l_0 norm. (a) $x_0 = 1$; (b) n=3.

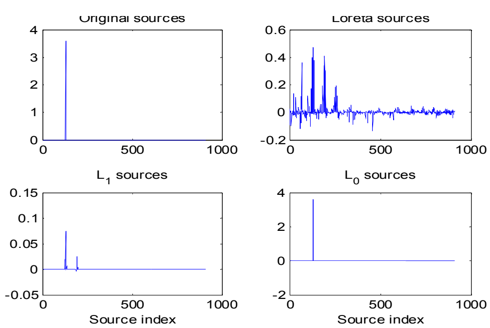
III. Results

D. Head model and parameters setup

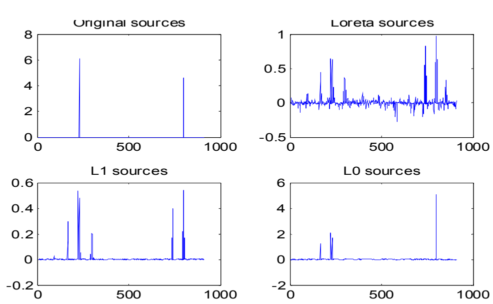
A 3-shell realistic head model is used for EEG source localization, whose conductivities for cortex, skull and scalp are $1.0\Omega^{-1}m^{-1}$, $1/80\Omega^{-1}m^{-1}$ and $1.0\Omega^{-1}m^{-1}$, respectively [9]. The solution space is restricted to cortical gray matter, hippocampus and other possible source activity areas, consisting of 910 cubic mesh voxels with 10 mm inter-distance. A standard 128 electrode system was registered on this head model. The lead field matrix A is calculated with charge model by BEM [10] for the 128 electrode system, and it is a matrix with dimension of 128×910. The detail for charge source model can refer to [11], [13]-[15]. The origin of the coordinate system is defined as the midpoint between the left and right pre-auriculars, and the directed line from the origin through the nasion defines the +*X*-axis, the +*Y*-axis is the directed line from the origin through the left pre-auricular. Finally, the +*Z*-axis is the line from the origin toward the top of the head (through electrode Cz). The punishment factor λ is 1.0, and n is set to be 2 with $2n$=4. The half cutoff frequency x_0 is adaptively adjusted in iterations following $x_0 = 0.03 \times \max(|X|)$, where X is the strengths of estimated sources in current iteration.

E. Simulated Results

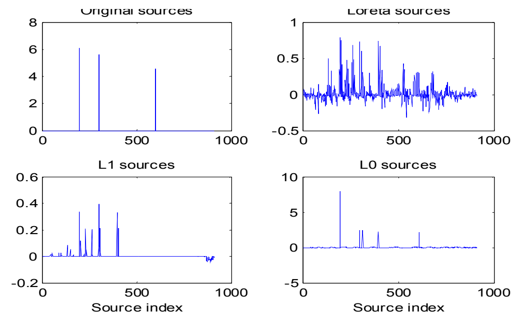
In this section, l_0 norm solution is compared with LORETA and l_1 solution using different source configurations. LORETA solution is regarded as the initial distribution for both l_0 and l_1 norm iterative procedures. Localization results of configurations with one, two and three sources are reported in Fig.2, respectively. Due to the paper limitation, we didn't show the source information on the realistic head model for 2-source configuration and 3-source configuration.



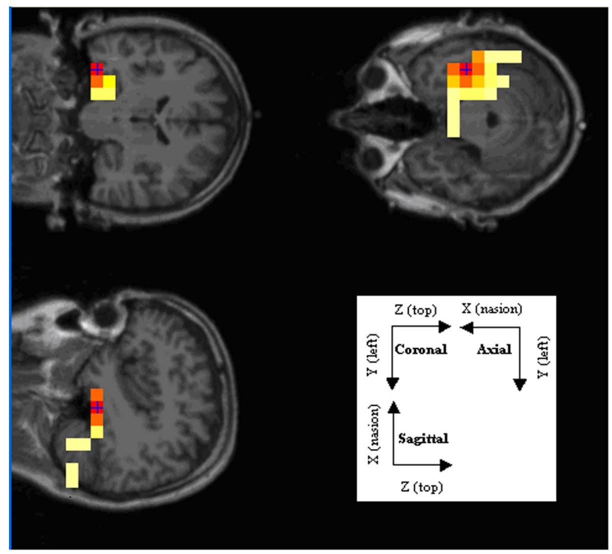
(a)



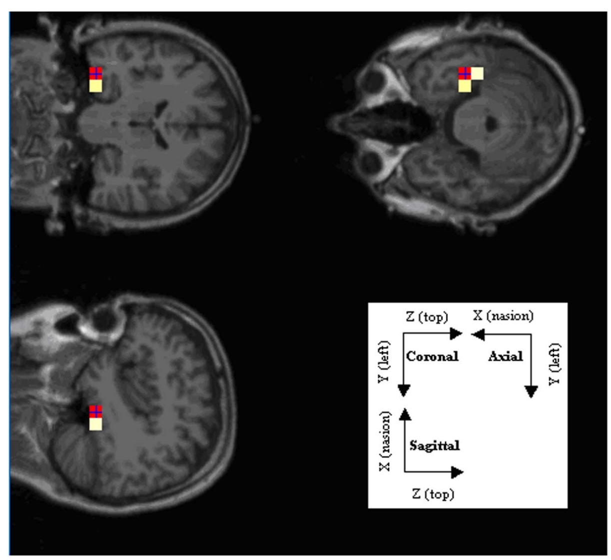
(b)



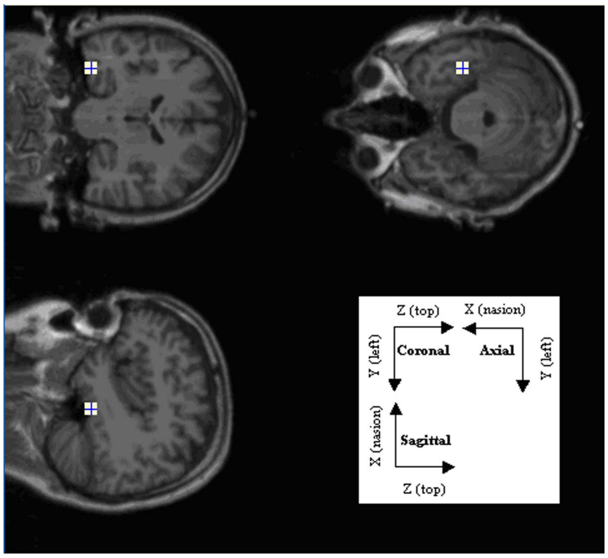
(c)



(x, y, z)=(-13. 0, -41. 0, -28. 3)
(d)



(x, y, z)=(-13. 0, -41. 0, -28. 3)
(e)



(x, y, z)=(-13.0, -41.0, -28.3)

(f)

Fig.2. LORETA, l_0 and l_1 solutions for simulated configurations with one, two and three sources. (a)~(c) are the information for actual sources and estimated sources listed by solution space index. (a) One source at (-13.0, -41.3, -28.3); (b) Two sources at (-73.0, -21.0, 31.7) and (-3.0, 39.0, 61.7); (c) Three sources at (7.0, -31.0, -28.3), (-73.0, -11.0, 41.7) and (37, 9.0, -18.3). (d)~(f) are the display of the source configuration in (a) on the realistic head model for LORETA, l_1 norm and l_0 solutions, respectively. Colorful rectangle area in (d)~(f) is the estimated source location; the blue cross line within the colorful rectangle area indicates the overlapping area of the simulated source and the estimated source.

IV. Discussion and Conclusion

The reported results showed that LORETA localized sources with a blurred distribution, and l_1 and l_0 solutions are more focal and suitable for localization of sparse sources. However, some difference between l_1 and l_0 solutions can be observed. l_1 solution failed to localize one source in the 2-source configuration, moreover l_1 solution is not clear with some strong fake sources still observed. However, l_0 solution basically recovered the spatial information of sources though the strengths are not consistent with the simulated ones. For all the three solutions, the localization performance will be lowered when source configuration becomes more complex with increasing number of sources. The current preliminary studies were implemented in the noise free conditions, and λ is a regularized parameter that can suppress the noise effect when noise is induced in recordings. But determination of suitable λ is an open issue, which needs to use some complicated regularization techniques like L-Curve, etc [12].

These preliminary results reveal that l_0 solution may be a promising technique for sparse EEG source localization. Compared to l_1 norm constraint, the new approximation function is much closer to the actual l_0 norm, therefore the new constraint can improve the source estimation performance. Certainly, both l_1 and l_0 solutions are not suitable for the estimation of extended sources, which LORETA alternatively is more competitive. How noise and parameters in the approximation function G(x), i.e. filter order n and cutoff frequency x_0, influence source estimation need to be further studied in future work.

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