31st Annual International Conference of the IEEE EMBS
Minneapolis, Minnesota, USA, September 2-6, 2009

1-D Steady State Analysis of a Two-equation Coupled System for Determination of Tissue Temperature in Liver during Radio Frequency Ablation

Tingying Peng, David P. O'Neill, Stephen J. Payne
Department of Engineering Science, University of Oxford, UK

***Abstract*— An analytical solution is provided for a two-equation coupled model for determination of liver tissue temperature during radio frequency ablation in the steady state with one-dimension in space[1]. Both analytical analysis and model simulation were conducted to investigate the effects of two crucial system parameters: blood perfusion rate and convective heat transfer coefficient on the tissue temperature field. The quantitative criteria were also derived, under which the two-equation coupled system can be approximated to a conventional single bio-heat equation system such as the Pennes model[2].**

I. INTRODUCTION

Radio Frequency Ablation (RFA) has been used for the treatment of focal primary and secondary liver malignancies as a minimally invasive, image-guided alternative to standard surgical resection ([2]). The classical bio-heat equation ([3]) has been widely used to model the electrical-thermal heating process during the ablation procedure. However, the convective heat transfer term between tissue and blood in the equation is oversimplified, assuming the blood as a volumetric sink of heat and is uniformly distributed throughout the tissue ([4], [5]). This inaccurate convection term would greatly affect the prediction of tissue temperature and thus the necrosis zone after the intervention, especially for highly blood perfused tissues such as liver.

A new mathematical model consisting of two coupled partial differential equations has been proposed for determining temperatures in living liver tissue ([1]). This model breaks an arbitrary control volume down to two subvolumes: tissue and blood, and sets up one bio-heat equation for each volume, similar to the approach of [6]. Heat is convectively transferred between the two subvolumes, making the two equations coupled. Unlike the previous model of [6], numerical solutions are calculated by solving both equations simultaneously, without further simplification of the model into a single heat equation.

This paper, alternatively, will present analytical solutions of the two-equation coupled system in steady state with one-dimensional spatial derivatives, followed by a sensitivity analysis to investigate how the two crucial parameters, blood flow perfusion and convective heat transfer coefficient, influence the tissue temperature field.

[1]Original model is presented in a parallel paper, [1].

[2]The research leading to these results has received funding from the European Community's Seventh Framework Programme under grant agreement n 223877, project IMPPACT.

II. THEORY

A. Model Equation

For a control volume V, tissue temperature T_t and blood temperature T_b are given by the following coupled bioheat equations ([1]):

$$\rho_t c_t V_t \frac{\partial T_t}{\partial t} = V_t \sigma_t |\vec{E}|^2 + V_t \nabla \cdot (k_t \nabla T_t) - U(T_t - T_b) \quad (1)$$

$$\rho_b c_b V_b \frac{\partial T_b}{\partial t} = V_b \sigma_b |\vec{E}|^2 + V_b \nabla \cdot (k_b \nabla T_b) + U(T_t - T_b) - \rho_b c_b (\nabla T_b \cdot \vec{u}) \quad (2)$$

in which V_t and V_b are tissue subvolume and blood subvolume respectively. Obviously, they satisfy:

$$V_t + V_b = V. \quad (3)$$

After non-dimensionlising temperature, $T_t{}' = \frac{T_t - T_0}{T_0}$, $T_b{}' = \frac{T_b - T_0}{T_0}$ (T_0 is the initial tissue and blood temperature at $t = 0$ and $x = 0$), time, $t' = \frac{t}{\tau}$, and space, $x' = \frac{x}{L}$, we rewrite bioheat equation in 1D:

$$\frac{\partial T_t{}'}{\partial t'} = \pi_1 |\vec{E}|^2 + \pi_2 \frac{\partial^2 T_t{}'}{\partial x'^2} - \frac{\pi_3}{1 - \pi_5}(T_t{}' - T_b{}') \quad (4)$$

$$\frac{\partial T_b{}'}{\partial t'} = \pi_1 |\vec{E}|^2 + \pi_2 \frac{\partial^2 T_b{}'}{\partial x'^2} + \frac{\pi_3}{\pi_5}(T_t{}' - T_b{}') - \frac{\pi_4}{\pi_5}\frac{\partial T_b{}'}{\partial x'}, \quad (5)$$

in which

$$\pi_1 = \frac{\tau \sigma}{\rho c T_0}, \pi_2 = \frac{\tau k}{\rho c L^2}, \pi_3 = \frac{\tau U}{\rho c}, \pi_4 = \frac{\tau u_x}{L}, \pi_5 = \frac{V_b}{V}. \quad (6)$$

Here we have made further approximation that $k_t = k_b = k$, $\rho_t = \rho_b = \rho$, $c_t = c_b = c$. For simplification yet without losing generality, we set up time constant $\tau = \rho c T_0$, thus

$$\pi_1 = 1, \pi_2 = \frac{k T_0}{\sigma L^2}, \pi_3 = \frac{U T_0}{\sigma}, \pi_4 = \frac{u_x \rho c T_0}{\sigma L}, \pi_5 = \frac{V_b}{V}. \quad (7)$$

B. Model Analysis

In steady state, Equation 4 turns into two coupled second-order ordinary differential equations. Using a spatial Laplace transform, we obtain:

$$s^2 \cdot \pi_2 \hat{T_t}' - \frac{\pi_3}{1-\pi_5}(\hat{T_t}' - \hat{T_b}') + \pi_1 |\hat{\vec{E}}|^2 = 0 \tag{8}$$

$$s^2 \cdot \pi_2 \hat{T_b}' - s \cdot \frac{\pi_4}{\pi_5}\hat{T_b}' + \frac{\pi_3}{\pi_5}(\hat{T_t}' - \hat{T_b}') + \pi_1 |\hat{\vec{E}}|^2 = 0 \tag{9}$$

By eliminating one variable, $\hat{T_b}'$, we can obtain the transfer function, H, which links radio frequency input power $|\vec{E}|^2$ with system output, T_t

$$H(s) = \frac{\hat{T}_t}{|\hat{\vec{E}}|^2} = -\frac{1}{\pi_2}\frac{s^2 - \frac{\pi_4}{\pi_2\pi_5}\cdot s - \frac{\pi_3\pi_4}{\pi_2\pi_5(1-\pi_5)}}{s^4 - \frac{\pi_4}{\pi_2\pi_5}\cdot s^3 - \frac{\pi_3}{\pi_2\pi_5(1-\pi_5)}\cdot s^2 + \frac{\pi_3\pi_4}{{\pi_2}^2\pi_5(1-\pi_5)}s}, \tag{10}$$

The static tissue temperature transfer function has numerator of order 2 and denominator of order 4. By using Taylor Expansion, an analytical form of zeros and poles of $H(s)$ (roots of numerator and denominator respectively) can be obtained. By making certain assumption of parameter values which we outline below, some zeros offset poles and thus reduce the system to a lower order. Details of the mathematical derivation will be presented in the Appendix, only a summary will be given here.

Assumption A: In the case of high blood perfusion relative to convection coefficient, i.e. $\frac{{\pi_4}^2(1-\pi_5)}{\pi_2\pi_3} \gg 1$, which is $\frac{\rho^2 c^2 {u_x}^2}{kU} \gg 1$ ($\pi_5 = \frac{V_b}{V} \ll 1$), $H(s)$ can be approximated into a second order system:

$$H_h(s) = \frac{\hat{T}_t}{|\hat{\vec{E}}|^2} = -\frac{1}{\pi_2}\frac{1}{s^2 - \frac{\pi_3}{\pi_2(1-\pi_5)}}. \tag{11}$$

substitute Equation 11 into Equation 8, thus $\hat{T_b}' = 0$, which means $T_b(x) = T_0 = T_b(0)$ - blood flow is quick enough to take away convective heat transfer from tissue immediately, thus keeping a constant temperature over space. The blood with constant temperature plays a role of a homogeneous sink, which is exactly the assumption of the Pennes model. In other words, the two-coupled equation can be simplified to the Pennes model for conditions of large blood flow perfusion and relatively small convective heat transfer coefficient between blood and tissue.

Assumption B: low blood perfusion relative to convection coefficient, i.e. $\frac{{\pi_4}^2(1-\pi_5)}{\pi_2\pi_3} \ll 1$, which is $\frac{\rho^2 c^2 {u_x}^2}{kU} \ll 1$ ($\pi_5 \ll 1$), $H(s)$ can also be reduced to a second order system:

$$H_l(s) = \frac{\hat{T}_t}{|\hat{\vec{E}}|^2} = -\frac{1}{\pi_2}\frac{1}{s(s - \frac{\pi_4}{\pi_2})}. \tag{12}$$

Substitute Equation 12 into Equation 8 to obtain $T_b(x) = T_t(x)$, blood temperature stays the same as tissue temperature over the space. $H_l(s)$ is also the transfer function governing spatial distribution of blood temperature. In this case, there is no need to break one control volume down to two subvolumes.

In two extreme cases, zero convection coefficient ($\pi_3 = 0$) and zero blood perfusion ($\pi_4 = 0$), Equation 11 and Equation 12 can both be further simplified to:

$$H'(s) = -\frac{1}{\pi_2}\frac{1}{s^2}. \tag{13}$$

which is the form of one-phase bioheat equations with only conductive heat transfer (no convective heat transfer between tissue and blood).

III. MODEL SIMULATION

All the values of parameters used in the model are shown in table I except blood flow perfusion rate u_x and convection coefficient U, accurate values of which are unknown in the literature. The effect of variations of these two parameters on the system behaviour will be investigated below.

Parameter and Description	Value and Unit
ρ,tissue/blood density	$1060 kgm^{-3}$
c, tissue/blood specific heat capacity	$3800 Jkg^{-1}K^{-1}$
k, tissue/blood thermal conductivity	$0.49 Wm^{-1}K^{-1}$
$\frac{V_b}{V}$, blood volume fraction	$0.02 m^0$
σ, electrical conductivity	$0.15 Sm^{-1}$
T_0, tissue/blood temperature at $x = 0$	$310K$
L, liver characteristic length	$0.1m$

TABLE I: Parameters used in the model simulations

By using the parameter values given in Table I, the values of the non-dimensional parameters in the transfer function are found to be:

$$\pi_1 = 1, \pi_2 = 1.03 \times 10^5, \pi_3 = 2.1 \times 10^3 U,$$
$$\pi_4 = 7.89 \times 10^{10} u_x, \pi_5 = 0.02.$$

Figure 1 shows spatial distribution of input radio frequency power used in the simulation.

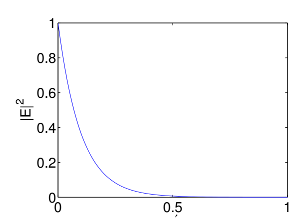
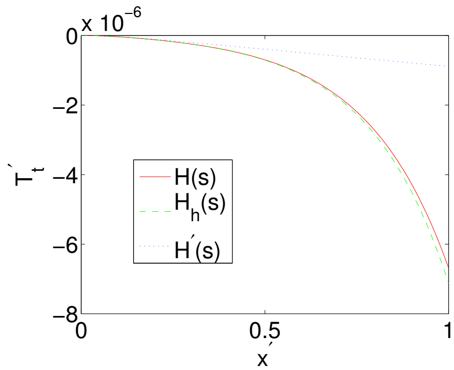


Fig. 1: Input radio frequency power $|E|^2 = e^{-10x'}$

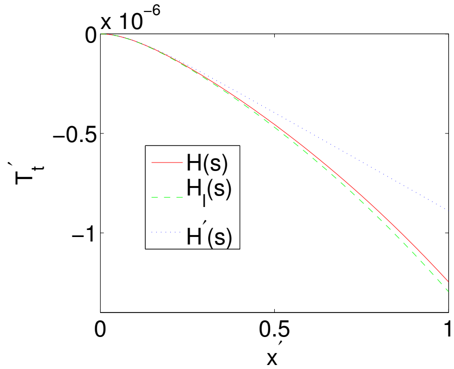
Figure 2(a) and Figure 2(b) plot model-predicted static spatial distribution of tissue temperature ${T_t}'$ in case of relative high blood flow perfusion ($u_x = 10^{-4}, U = 1000, \frac{\rho^2 c^2 {u_x}^2}{kU} = 297 \gg 1$) and low flow perfusion respectively ($u_x = 10^{-6}, U = 1000, \frac{\rho^2 c^2 {u_x}^2}{kU} = 0.0297 \ll 1$).[3] Both the responses of simplified transfer functions,

[3]Note the values shown in the figure are relative temperature value, ${T_t}'$, in reference to temperature at $x = 0$, T_0. That's why it is negative as temperature in other region is lower than T_0.

$H_h(s)$ and $H_l(s)$, closely approximate the response of the original system, $H(s)$. However, the output of uncoupled system which models only conduction heat transfer, $H'(s)$, is obviously different from the original system, suggesting that tissue/blood convective heat transfer plays an essential role in the radio frequency ablation process.



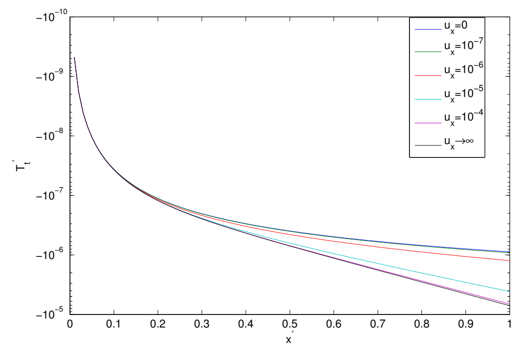
(a) high blood perfusion



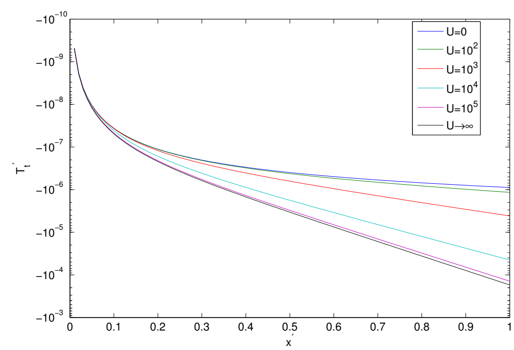
(b) low blood perfusion

Fig. 2: Model-predicted static spatial distribution of tissue temperature field $T_t{}'$ in response to input radio frequency power shown in Figure 1

Figure 3(a) shows how spatial tissue temperature distribution varies with different levels of blood perfusion, with fixed convective heat transfer coefficient $U = 1000$. The top line for $u_x = 0$ is the response of $H'(s)$ (no heat transfer between tissue and blood) while the bottom line for $u_x \to \infty$ is the response of $H_h(s)$ ($\frac{\rho^2 c^2 u_x{}^2}{4kU} \to \infty$). The figure shows that tissue temperature was found to decrease with increasing blood flow perfusion, however, this temperature drop no longer happen with further increase of u_x when it reach a certain level, suggested by the almost identical lines of $u_x = 10^{-3}$ and $u_x \to \infty$. This implies that heat transfer between tissue and blood is now primarily limited by the other factor, tissue/blood convective heat transfer coefficient. Figure 3(b), in contrast, plots spatial tissue temperature variation at different heat transfer coefficients, with fixed blood perfusion u_x. Similar to figure 3(a), the top line for $U = 0$ also represent the response of $H'(s)$, however, the bottom line for $U \to \infty$ is now the response of $H_l(s)$ ($\frac{\rho^2 c^2 u_x{}^2}{4kU} \to 0$). In figure 3(b), a temperature drop was also found with increasing U, reflecting the effect of increased convective heat transfer between tissue and blood. More interestingly, a similar upper-limit effect was also found for U, illustrating that the heat transfer now primarily limited by blood flow perfusion rate u_x. This demonstrates that the effective convective heat exchange between tissue and blood is due to the coordination of blood perfusion and convection transfer coefficient.



(a) variation of u_x



(b) variation of U

Fig. 3: Spatial tissue temperature distribution at various blood perfusion u_x and convective heat transfer coefficient U.

IV. CONCLUSION

This paper provided the analytical solution of a two-equation coupled model for determination of liver tissue temperature during radio frequency ablation in the steady state with one-dimension in space. Sensitivity analysis was conducted to carefully examine the effects of two crucial system parameters, blood perfusion rate and convective heat transfer coefficient on the tissue temperature field. It has been demonstrated by both analytical analysis and model simulation that the original two-equation coupled system can be approximated by a single bio-heat equation under following conditions:

1) in the condition of high blood perfusion and low convective heat transfer coefficent, blood temperature stays constant over space and the system is simplified into the Pennes model;

2) in the condition of low blood perfusion and high convective heat transfer coefficient, blood temperature is equivalent to tissue temperature over space and the two bio-heat

equations for blood and tissue subvolumes are essentially the same.

Appendix

Rewrite transfer function $H(s)$ as:

$$H(s) = -\frac{1}{\pi_2}\frac{s^2 - A\cdot s - B}{s^4 - A\cdot s^3 - B\cdot s^2 + AB\pi_5 s} \tag{14}$$

where $A = \frac{\pi_4}{\pi_2\pi_5}$ and $B = \frac{\pi_3}{\pi_2\pi_5(1-\pi_5)}$.

Equation 14 can be converted into zero-pole-gain form as:

$$H(s) = \frac{\hat{T}_t}{|\hat{\vec{E}}|^2} = -\frac{1}{\pi_2}\frac{(s-n_1)(s-n2)}{s(s-d_1)(s-d_2)(s-d_3)} \tag{15}$$

Two zeros (roots of numerator) are quite straightforward:

$$n_{1,2} = \frac{A}{2} \pm \frac{\sqrt{A^2+4B}}{2} \tag{16}$$

Three poles d_1, d_2, d_3 are roots of cubic equation:

$$s^3 - A\cdot s^2 - B\cdot s + AB\pi_5 = 0 \tag{17}$$

The solution of Equation 17 is based on Cardano's method.

$$\begin{aligned} q &= -\frac{1}{9}(A^2+3B) && (18)\\ r &= \frac{1}{27}A^3 + \frac{1}{6}AB - \frac{\pi_5}{2}AB && (19)\\ \Delta &= q^3 + r^2 \\ &\approx -\frac{1}{108}B^2(A^2+4B) - \pi_5(\frac{1}{27}A^4B + \frac{1}{6}A^2B^2) && (20) \end{aligned}$$

as $\pi_5 = 0.02$, suggesting blood subvolume a small fraction of total control volume. Since $\Delta < 0$, all three roots, d_1, d_2, d_3, are real.

With the assumption of $\frac{A^2}{B} \ll \frac{1}{\pi_5}$, we have $\pi_5(\frac{1}{27}A^4B + \frac{1}{6}A^2B^2) \ll \frac{1}{108}B^2(A^2+4B)$, thus by using Taylor expansion:

$$\sqrt{-\Delta} \approx \frac{B\sqrt{A^2+4B}}{6\sqrt{3}} + \frac{\sqrt{3}\pi_5 A^2}{\sqrt{A^2+4B}}(\frac{A^2}{9} + \frac{B}{2}) \tag{21}$$

Defining $s = \sqrt[3]{r + i\sqrt{-\Delta}}$ and $t = \sqrt[3]{r - i\sqrt{-\Delta}}$, the solutions are:

$$\begin{aligned} d_1 &= s + t + A/3 \\ &\approx \frac{A}{2} + \frac{\sqrt{A^2+4B}}{2} + \frac{\pi_5 A^2}{2\sqrt{A^2+4B}} - \frac{\pi_5 A}{2} && (22)\\ &\approx n_1 \\ d_2 &= -\frac{1}{2}(s+t) + A/3 + \frac{\sqrt{3}}{2}(s-t)i \\ &\approx \frac{A}{2} - \frac{\sqrt{A^2+4B}}{2} - \frac{\pi_5 A^2}{2\sqrt{A^2+4B}} - \frac{\pi_5 A}{2} && (23)\\ &\approx n_2 \\ d_3 &= -\frac{1}{2}(s+t) + A/3 - \frac{\sqrt{3}}{2}(s-t)i \\ &\approx \pi_5 A && (24) \end{aligned}$$

As a result, transfer function $H(s)$ is approximated to

$$H_l(s) = -\frac{1}{\pi_2}\frac{1}{s(s-\pi_5 A)} = -\frac{1}{\pi_2}\frac{1}{s(s-\frac{\pi_4}{\pi_2})}, \tag{25}$$

which is Equation 12.

On the other hand, when we make the assumption that $\frac{A^2}{B} \gg 1$, equivalent to $\frac{B}{A^2} \ll 1$, again, we use Taylor expansion and obtain:

$$\Delta \approx -\frac{\pi_5 A^4 B}{27}. \tag{26}$$

Follow the same step presented above, the solutions can be obtained as:

$$\begin{aligned} d_1 &\approx \frac{A}{2} + \frac{\sqrt{A^2+4B}}{2} - \pi_5\frac{B}{A} \approx n_1 && (27)\\ d_{2,3} &\approx \frac{A}{4} - \frac{\sqrt{A^2+4B}}{4} + \frac{\pi_5}{2}\frac{B}{A} \pm \sqrt{\pi_5 B} \\ &\approx -\frac{B}{2A} \pm \sqrt{\pi_5 B} && (28) \end{aligned}$$

After making stricter limitation $\frac{A^2}{B} \gg \frac{1}{pi_5}$, we have $\frac{B}{A} \ll \sqrt{\pi_5 B}$, so the approximated transfer function is:

$$\begin{aligned} H_h(s) &= -\frac{1}{\pi_2}\frac{s - \frac{B}{A}}{s(s + \frac{B}{2A} + \sqrt{\pi_5 B})(s + \frac{B}{2A} - \sqrt{\pi_5 B})} \\ &\approx -\frac{1}{\pi_2}\frac{1}{s^2 - \pi_5 B} = -\frac{1}{\pi_2}\frac{1}{s^2 - \frac{\pi_3}{\pi_2(1-\pi_5)}} && (29) \end{aligned}$$

which is Equation 11.

References

[1] D. P. O'Neill, T. Peng, and S. J. Payne, "A two-equation coupled system for determination of tissue temperature in liver during radio frequency ablation." *Submitted to 31st Annual International Conference of the IEEE Engineering in Medicine and Biology Society*, April 2009.

[2] E. J. Berjano, "Theoretical modeling for radiofrequency ablation: state-of-the-art and challenges for the future." *Biomed Eng Online*, vol. 5, 2006.

[3] H. H. Pennes, "Analysis of tissue and arterial blood temperatures in the resting human forearm," *J Appl Physiol*, vol. 85, no. 1, pp. 5–34, July 1998.

[4] C. K. Charny, *Advances in Heat Transfer Bioengineering Heat Transfer*. San Diego: Academic Press, Inc., 1992, vol. 22, pp. 19–155.

[5] H. Arkin, L. X. Xu, and K. R. Holmes, "Recent developments in modeling heat transfer in blood perfused tissues," *Biomedical Engineering, IEEE Transactions on*, vol. 41, no. 2, pp. 97–107, 1994.

[6] M. M. Chen and K. R. Holmes, "Microvascular contributions in tissue heat transfer." *Annals of the New York Academy of Sciences*, vol. 335, pp. 137–150, 1980.