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Bifurcations in Morris-Lecar Model Exposed to DC Electric Field

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***Abstract*— As an important neuron model, the Morris-Lecar (ML) equations can exhibit classes I and II excitabilities with appropriate system parameters. In this paper, the effects of external DC electric field on the neuro-computational properties of ML model are investigated using bifurcation analysis. We obtain the bifurcation diagram in two dimensional parameter space of externally applied DC current and trans-membrane potential induced by external DC electric field. The bifurcation sets partition the two dimensional parameter space about the qualitatively different behaviors of the ML model. Thus the neuron's information encodes the stimulus information, and vice versa, which is significant in neural control. Furthermore, we identify the electric field as a key parameter to control the transitions among four different excitability and spiking properties.**

I. INTRODUCTION

The type of bifurcation the neuron experiences determines the neuronal excitable and spiking properties, and hence the neuro-computational attributes [1]. Hodgkin suggested two different classes of neurons, namely Class I and Class II neurons according to their frequency responses to a constant current stimulation [2]. For the bifurcation theory, Class I excitability is related to a saddle-node bifurcation, and action potentials are generated with arbitrarily low frequency, depending on the strength of the applied current. On the other hand, Class II excitability is concerned with the Hopf bifurcation, and Action potentials are generated in a certain frequency band that is relatively insensitive to changes in the strength of the applied current [1, 3]. Many mathematical models have been proposed to describe neural activities [1, 4], which may exhibit either or both of two firing modes. For example, the original Hodgkin-Huxley [5] and FitzHugh-Nagumo model (FHN) [6] are the typical Class II neuron models. The Hindmarsh-Rose model [7, 8], the Morris-Lecar model [9], the Wilson model [10], and the Izhikevich model [4] show Class I excitabilities. The two- or three-dimensional models may have advantages not only in their practicality in simulations with large scale coupled neuronal systems but also in its clarity of mathematical essence of bifurcation structure. Because the ML model has biophysically meaningful and measurable parameters, the model became quite popular in computational neuroscience community [11, 12].

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Due to the rapidly increased electromagnetic exposure in environment [13], many diseases probably caused by electromagnetic exposure are reported [14, 15]. Thus the interaction between electric field and biological tissues has been of longstanding interest. The effect of externally applied DC electric fields on neural excitability has been demonstrated in many neuronal systems [16-20]. These studies show that DC electric fields could decrease or increase neuronal excitability [16], modulate neuronal thresholds [17] and neural firing [19], and cause long time polarization [20]. However, it remains unclear that how the DC electric fields affect the neuro-computational properties of neurons. Especially, to the best of our knowledge, this problem has not been addressed using a bifurcation analysis.

In [21], we have investigated the Hopf bifurcations in HH model caused by external DC electric field. In this paper, we will give a detailed bifurcation analysis to further examine the role of electric field in determining the neuro-computational properties in context of ML model. Based on the results we obtained in [21], the modified ML model exposed to DC electric field is proposed. In the modified ML model a new parameter V_{E} is introduced to denote the effect of external electric field. We calculate the bifurcations in the modified ML model in two dimensional parameter space $I_{\mathrm{ext}} - V_{\mathrm{E}}$. The parameter space are partitioned into qualitatively different regions by bifurcation curves. Thus we relate the qualitatively different behaviors of neuron to the external stimuli, which is useful in neuron control. This study maybe throw some light on the interference mechanism between biological systems and electric exposure.

II. MODELS

A. Morris-Lecar model

The ML model is composed of the following differential equations:

$$C_{\mathrm{M}}\frac{dV}{dt} = I_{\mathrm{ext}} - [\overline{g}_{\mathrm{Ca}}M_{\infty}(V - V_{\mathrm{Ca}}) + \overline{g}_{\mathrm{K}}N(V - V_{\mathrm{K}}) + g_{\mathrm{L}}(V - V_{\mathrm{L}})] \tag{1a}$$

$$\frac{dN}{dt} = \phi\frac{N_{\infty} - N}{\tau_N} \tag{1b}$$

where V represents membrane potential in mV, the time t is measured in msec, the variables $N \in [0, 1]$ represents the potassium activation varable. The Parameters V_{Ca}, V_{K} and V_{L} represent the equilibrium potentials of calcium, potassium and leak current, respectively. They are determined uniquely by the Nernst's equations. $\overline{g}_{\mathrm{Na}}$, $\overline{g}_{\mathrm{K}}$ and g_{L} are the maximal conductance of the corresponding ionic currents. They reflect the ionic channel density distributed over the membrane. C_{M}

is the membrane capacitance, I_{ext} is the externally-applied DC current. The parameter ϕ sets the time scale for the recovery process. The steady state activation M_∞ and N_∞ are nonlinear functions of V, given by the following equations:

$$M_\infty = 0.5\left[1 + \tanh\left(\frac{V - V_1}{V_2}\right)\right]$$
$$N_\infty = 0.5\left[1 + \tanh\left(\frac{V - V_3}{V_4}\right)\right]$$

where V_1 and V_3 are the the activation midpoint potential at which the corresponding currents are half activated. V_2 and V_4 denote the slope factor of the activation. The time constant τ_N about the potassium activation is described by

$$\tau_N = \frac{1}{\cosh\left(\frac{V-V_3}{2V_4}\right)} \tag{2}$$

B. The modified ML model exposed to external electric field

According to our analysis in literature [21], when exposed to DC electric field E, the original ML model is modified as the following form:

$$\begin{aligned} C_{\mathrm{M}}\frac{dV}{dt} &= I_{\mathrm{ext}} - [\overline{g}_{\mathrm{Ca}}M_\infty(V + V_{\mathrm{E}} - V_{\mathrm{Ca}}) \\ &\quad + \overline{g}_{\mathrm{K}}N(V + V_{\mathrm{E}} - V_{\mathrm{K}}) + g_{\mathrm{L}}(V + V_{\mathrm{E}} - V_{\mathrm{L}})] \quad (3a) \\ \frac{dN}{dt} &= \phi\frac{N_\infty - N}{\tau_N} \quad (3b) \end{aligned}$$

The new parameter V_{E} is the induced trans-membrane potential which reflects the effect of the electric field E and can be calculated as given in [21].

V_{E} behaves as an electromotive force added to the membrane. V_{E} does not change the basic structure of ML model but to change the anti-electromotive forces of calcium current, potassium current and leak current. According to control theory, the introduction of V_{E} could be regarded as a disturbance applied to the original system and its dynamic performance under disturbance should be investigated.

Throughout this paper, except for I_{ext} and V_{E}, all the other parameters involved in ML model are fixed as values for a class I neuron model as given in [22].

III. DEFINITION AND DETECTION OF BIFURCATIONS

A bifurcation is a change of qualitative behavior of a dynamical system at special values of the parameters. The codimension of a bifurcation is the minimum dimension of a parameter space in which the bifurcation may happen in a persistent way [23].

A. Codimension one bifurcations

Hopf bifurcation (H) A continuous fundamental path of an equilibrium point loses its stability as it intersects a secondary path of a periodic solution. The location of a Hopf bifurcation on the equilibrium point is characterized by a complex conjugate pair of linear eigenvalues of the Jacobian matrix whose real part passes through zero. When the secondary path is stable, it is the supercritical Hopf bifurcation (sH). Conversely, when the secondary path is unstable, it is the subcritical Hopf bifurcation (uH) .

Saddle-node bifurcation (sn) Two equilibrium points coalesce and disappear. At this bifurcation point, the Jacobian matrix of the equations at the equilibrium point has a zero eigenvalue.

Double cycle or saddle-node of cycles (dc). Two periodic solutions with finite amplitude coalesce and disappear.

saddle loop or homoclinic bifurcation (sl) The amplitude of a periodic orbit may increase until it captures a saddle point and disappears its period tending to infinity as approaching bifurcation point.

B. Codimension two bifurcations

Three types of codimension two bifurcations as follows are considered.

Cusp (c) Three equilibrium points coalesce into one.

Takens-Bogdanov bifurcation (TB) The Jacobian matrix of the equations at the equilibrium point has two zero eigenvalues. On a two dimensional bifurcation diagram (2BD), TB locates on the sn curve, and it is the terminus of the homoclinic and the Hopf bifurcation curves tangent to the sn curve at this point.

Degenerate Hopf bifurcation (dH) The stability of the periodic solution which bifurcates at the Hopf bifurcation point changes. A dc curve is terminated at this point on a 2BD.

In this study, Numerical computations for the bifurcation diagrams are conducted using MATCONT, a new MATLAB continuation package for the study of parameterized ODE systems [24]. Matcont can detect several bifurcation points automatically and can trace both stable and unstable branches of equilibria and periodic solutions. Numerical integration of the system equations for obtaining the trajectories was conducted in MATLAB using ode45.

IV. RESULTS AND ANALYSIS

In this section, first we give a global structure of bifurcations in $I_{\mathrm{ext}} - V_{\mathrm{E}}$ two dimensinal parameter space, and then we analyze the bifurcations about I_{ext} for different values of V_{E} to show how the electric field affects the neuronal excitability and spiking properties.

Fig. 1 shows the two-parameter bifurcation diagram of the modified ML model. The abscissa and the ordinate are I_{ext} and V_{E}, respectively. Hopf (uH_1, uH_2, sH), saddle-node (sn_1, sn_2), double cycles (dc_1, dc_2) and homoclinic bifurcation (sl_2) curves are displayed as solid, dash-dotted, dashed and dotted lines, respectively. By these curves, Fig. 1 is then partitioned into six qualitatively different regions (A-F). Schematic phase portraits for each region are given. Stable equilibrium points are shown as solid dots, unstable ones are crosses, stable limit cycles are closed curves with solid lines, and unstable periodic orbits are dashed lines. Different phase portraits indicate qualitatively different behaviors of ML model.

In region A, the unique steady state is an stable equilibrium point (EP), and the ML behaves as an excitable membrane. In B, a periodic solution, i.e. a limit cycle (LC), is the unique stable steady state and an unstable equilibrium

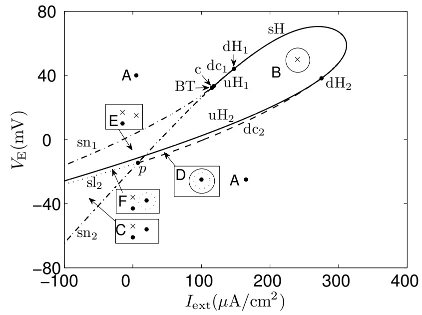

Fig. 1. Bifurcation diagram of ML model in $I_{\rm ext} - V_{\rm E}$ parameter space

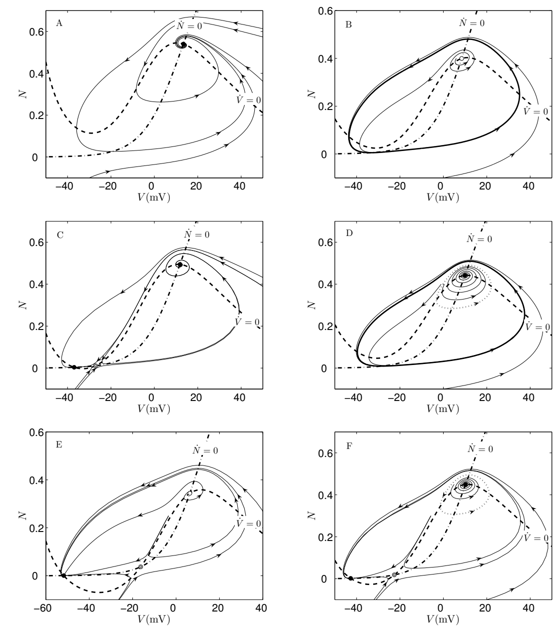

Fig. 2. Examples of various phase portraits in regions A-F in Fig. 1.

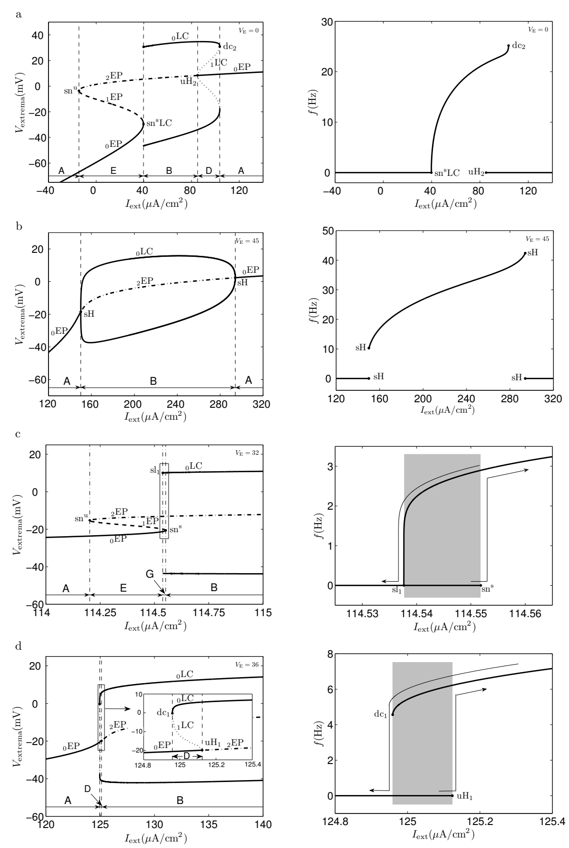

Fig. 3. Bifurcation diagram of ML model about $I_{\rm ext}$ with (a) $V_{\rm E}$ = 0mV, (b) $V_{\rm E}$ = 45mV, (c) $V_{\rm E}$ = 32mV and (d) $V_{\rm E}$ = 36mV. *left*: extremum of V, *right*: frequency. The neuron exhibits (a) Class I excitability and Class I spiking, (b) Class II excitability and Class II spiking without the bistability. (c) Class 2 excitability and Class I spiking and (d) Class II excitability and Class II spiking with the bistability, respectively.

point exist within it. In C, three equilibrium points coexist and two of them are stable, i.e. bistability of equilibrium points. In D, two stable attractors, an equilibrium point and a periodic solution, coexist with one unstable limit cycle, i.e. bistability of an equilibrium point and a limit cycle. In E, three equilibrium points coexist but only one of them is stable. Region F is another bistability region of equilibrium points, but there is also an unstable limit cycle. The typical responses of phase portraits for different regions are shown in Fig. 2.

To illustrate how the electric field affects the excitability of neurons, we calculate bifurcation diagram about $I_{\rm ext}$ for different values of $V_{\rm E}$, and show that the single parameter $V_{\rm E}$ can control the transitions among four different excitability and spiking properties.

For $V_{\rm E}$ = 0, the ML system shows a Class I excitability and spiking feature by varying $I_{\rm ext}$. As shown in Fig. 3a, the existence of sn$^{\rm s}$LC (saddle-node on limit cycle) is essential for the occurrence of Class I excitability and spiking, and the bifurcating state shows zero frequency response. For $V_{\rm E}$ = 45, the transition between excitability and spiking is supercritical Hopf (sH) bifurcation, and the bifurcating state holds a certain nonzero frequency. So the ML neuron behaves Class II excitability and spiking (Fig. 3b). When $V_{\rm E}$ is fixed at the value of 32, we can observe Class II excitability and Class I spiking in Eq. (1). As shown in Fig. 3c, as $I_{\rm ext}$ increases, the system state keeps silent and moves into the narrow region G since the sl bifurcation does not affect the existing equilibrium point. With more increment of $I_{\rm ext}$, the stable equilibrium disappears via sn, then we have spiking with a nonzero frequency value in region B. This process results in Class II excitability. In the inverse process, since the spiking is not related with the saddle-node bifurcation of equilibrium points, the frequency of the spiking vanishes gradually as the system approaches the sl point. Then the spiking is terminated and we have a quiescent state. Hence a Class II excitable and Class I spiking process is demonstrated. Finally when $V_{\rm E}$ = 36, the ML model shows Class II excitability and Class II spiking with the bistability. The uH and the dc bifurcations form a narrow region (region D) where bistability occurs.

V. CONCLUSIONS

The aim of this paper is to investigate the effects of external DC electric field on the neuro-computational properties in context of ML model using bifurcation analysis. The obtained bifurcation curves partition the two dimensional $I_{\mathrm{ext}} - V_{\mathrm{E}}$ parameter space into different regions according to qualitatively different firing patterns of the ML neuron. The stimulus parameters can be quantified by the various neuronal behaviors. On the one hand, one can interpret neuron activity as a representation of the parameter values. On the other hand, one can determine the parameter values to make the neuron elicit certain specific action potential sequences, which is significant in neuron control. By varying the single electric field parameter V_{E}, we also obtain four primary neuronal excitability and spiking features which are associated with different bifurcation mechanism. The results give some hints for research on interference between electromagnetic exposure and biological systems.

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