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A New Model-based Estimation of Ellipses for Object Representation

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***Abstract*— Fitting geometric models to objects of interest in images is one of the most classical problems studied in computer vision field. As a result of its strong representation power and flexibility, conic is one of the geometric primitives widely used in a large number of image analysis applications, in practice. As opposed to most existing conic fitting methods minimizing the fitting error with the use of the second order polynomial representation, in this paper, we propose a new method that formulates the geometric fitting problem as a process of seeking for the optimal mapping to a bivariate normal distribution model. As a result, some critical disadvantages tightly coupled with those methods following the routine polynomial representation can be well overcome. To demonstrate this, a set of carefully designed comparison experiments is conducted to show the superiority of the newly proposed method to a representative method using the polynomial representation. Additionally, the practical effectiveness of the proposed method is further manifested using a set of real image data with a promising accuracy.**

I. INTRODUCTION

One fundamental yet challenging problem in computer vision is to use primitive models to represent image components in 2D image spaces. Given the fact that a 2D image is a perspective projection of a 3D view, a 2D abstract representation is only useful when it preserves some geometric properties of the objects in a 3D environment [1]. As conicity is such a property preserving geometric features when projecting a 3D object to a 2D image, conic segments that could be generated by the intersections of a plane with a double cone are widely used in many computer vision applications, such as the recognition of 3D objects [2], identification of traffic signs [3], and tracking of biological cells [4]. Partially due to the nonlinear nature of this problem [5], there still remains much room for improvement despite the intensive studies on this topic though.

A rich volume of studies on methods for fitting conics to data can be found in literature [6],[7],[8],[9]. Most of these algorithms describe conics with a second order polynomial representation that follows:

$$f(\Omega,\Theta) = \Omega^T \cdot \Theta = a\,\omega_x^2 + b\,\omega_x\omega_y + c\,\omega_y^2 + d\,\omega_x + e\,\omega_y + f = 0 \tag{1}$$

where $\Theta = \begin{pmatrix} a & b & c & d & e & f \end{pmatrix}^T$ is the coefficient vector to be estimated and $\Omega = \begin{pmatrix} \omega_x^2 & \omega_x\omega_y & \omega_y^2 & \omega_x & \omega_y & 1 \end{pmatrix}^T$ is a vector established with the x- and y-coordinate of a data point (ω_x, ω_y) in a 2D space.

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Given a set of points, the best estimates of parameters under the metric of Least Squared Error (LSE) are those that minimize the sum of squared algebraic distances:

$$\Theta^* = \arg \min_{\Theta \in S \subseteq \mathbb{R}^6} \sum_{i=1}^{N} f(\Omega_i, \Theta) \tag{2}$$

where S is a subset of $\mathbb{R}^6$ that contains all possible solutions to Θ.

Many ways of constructing the set S appear in the literature. One of the most common constraints is to normalize the parameter vector to unit length, i.e., $\|\Theta\| = 1$, since any scaled solution to (1) is still a solution. Other constraints imposed on the set S include $a+c=1$ and $f=1$, as proposed by Gander [6] and Rosin [8], respectively. More complicated relationships across the coefficients have been investigated by Bookstein [10], who proposes $a^2 + \frac{1}{2}b^2 + c^2 = 1$. Fitzgibbon, *et al.* [7] forces the resulting geometric shape to be ellipse-specific by adding the discriminant $b^2 - 4ac < 0$, and turns this inequality into an equivalent constraint in equation form as: $4ac - b^2 = 1$.

Of all those studies on the conic fitting problem, ellipses are often the conics of interest for object representation and image analysis. Although the polynomial representation is widely used for estimating the parameters of ellipses, several notable disadvantages are noticed. First of all, as certain constraints need to be stipulated to restrict to the intended ellipse the resulting geometric shape of the best fitting, e.g. limiting Θ to $S \subseteq \mathbb{R}^6$, it is a constrained optimization problem with a high computational complexity. Additionally, each data sample, depending on its position relative to the underlying ellipse, has different impact on the parameter estimation with the error metric defined in (2). This is also known as the "high-curvature" bias problem that is particularly evident when data is noisy [5],[10]. Furthermore, the geometric interpretation of the estimated parameters is not straightforward. For the plotting purpose, estimated parameters of the resulting ellipse need to be converted from the polynomial representation using complicated formulae.

To address these problems, we propose a new ellipse fitting mechanism that formulates the parameter estimation problem as a searching process for an optimal mapping to a Gaussian bivariate distribution model. In our experiments where data is coupled with noise, this model based approach presents a robust fitting performance partially in that it combines the idea of level set theory and parametric deformable model [11]. The performances of the new method are compared with

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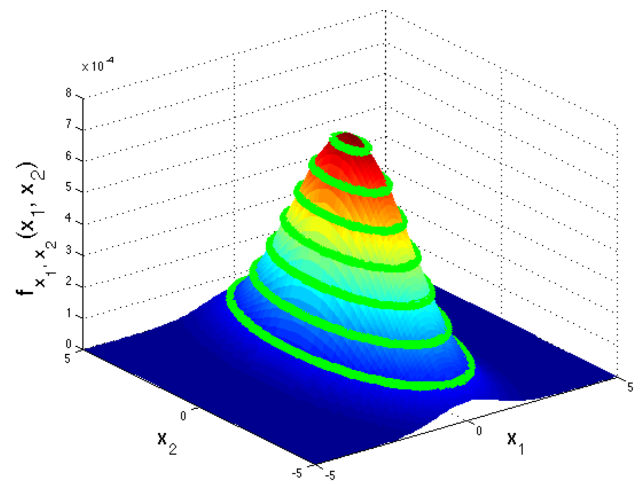

Fig. 1. A series of iso-contours with elliptical shapes (in green) are superimposed on a typical Gaussian bivariate probability density surface in a 3D space.

those of an existing method representative of those methods using the routine second order polynomial representation. Furthermore, we apply our method to a set of real image data to show its practical efficacy.

II. ELLIPTICAL FIT USING THE DEFORMABLE NORMAL DISTRIBUTION MODEL

This new ellipse fitting method is enlightened by the fact that the probability density surface of a Gaussian bivariate distribution can be deemed as a composition of a set of 2D contours, i.e., the iso-contours, as illustrated in Fig. 1. In general, these contours, shown in green in Fig. 1, are elliptical in shape, provided that the variances associated with the two random variables, i.e. x_1 and x_2, are not equal. Additionally, all such contours have distinct scales that are determined by their positions in "height" (i.e., along the vertical axis).

This important fact suggests that, given an arbitrary set of 2D points from a perfect underlying ellipse, we can imagine that these points are dispersed along an iso-contour on the probability density surface of a bivariate normal distribution. Apparently, such a surface consists of a family of 2D iso-contours of different scales. Deforming the surface and moving vertically along its "height" direction, we can find the ellipse that best fits to the given data based on a given criterion, such as the minimization of the least-square error. In this way, we formulate the geometrical model fitting problem as a problem where we seek for the best mapping relationship with a deformable statistical distribution model.

In probability theory, it is well known that a multivariate Gaussian Probability Density Function (PDF) can be mathematically described as:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi\,|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\right) \quad (3)$$

where $\mathbf{x}$ is a multivariate vector: $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_\ell \end{pmatrix}^T$. In our case, the number of variables is two, i.e., $\ell = 2$.

It can be observed that all those elliptical contours sitting on the bivariate normal density surface $\mathbb{N}(\mu,\Sigma)$ in a 3D space share the same factor before the exponential term in (3). As a result, an arbitrary elliptical contour can simply be represented using the exponential term only:

$$(\mathbf{x}-\mu)^T \mathbf{U}\Lambda^{-1}\mathbf{U}^T (\mathbf{x}-\mu) = c^2 \quad (4)$$

where we have $\Sigma = \mathbf{U}\Lambda\mathbf{U}^T$; Λ is a diagonal matrix and $\mathbf{U}$ is a unitary matrix; c is a constant representing the distance measured with the number of standard deviations from any point on the ellipse to the mean of the distribution, i.e., the Mahalanobis distance.

One immediate observation from (4) is that the location, scale and shape information of an ellipse characterized by (4) can now be readily represented by the mean μ, the constant c, and the covariance matrix Σ in a straightforward way. Let's denote $\chi = \{\mathbf{x}_i \,|\, \mathbf{x}_i = \begin{pmatrix} x_{i_1} & x_{i_2} \end{pmatrix}^T, i = 1,2,\ldots,N\}$ as coordinates of a set of data points to be fit with. The resulting parameter estimates can be solved by taking the estimation process discussed as follows.

(1) **Scale**: Given the representation in (4), the following step of this new method is to find the best scale that enables the resulting ellipse to get as close as possible to all the given data samples. This can be formulated as an optimization problem of minimizing a cost function established with the Mean Squared Error (MSE) metric:

$$(c^*)^2 = \arg\min_{c^2\in\mathbb{R}^+} J(c) \quad (5)$$

where $J(c) = \sum_{i=1}^{N}\left((\mathbf{x}_i-\mu)^T\,\Sigma^{-1}\,(\mathbf{x}_i-\mu) - c^2\right)^2$.

The optimal solution c^* minimizing $J(c)$ can be attained by setting to zero the derivative of $J(c)$ with respect to c^2. With simple differential computations, the analytical solution is as follows:

$$c^* = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i-\mu)^T\,\Sigma^{-1}\,(\mathbf{x}_i-\mu)} \quad (6)$$

(2) **Location**: The ellipse centroid can also be shifted towards its desired position by minimizing the following cost function:

$$J(\mu) = \|\mathbf{f}(\mu)\|^2 = \sum_{i=1}^{N} f_i(\mu)^2 \quad (7)$$

where $\mathbf{f}(\mu) = \begin{bmatrix} f_1(\mu) & f_2(\mu) & \cdots & f_N(\mu) \end{bmatrix}^T$ and $f_i(\mu)$ is defined as: $f_i(\mu) = (\mathbf{x}_i-\mu)^T\,\Sigma^{-1}\,(\mathbf{x}_i-\mu) - c^2$.

Since the cost function $J(\mu)$ is a nonlinear function of μ, Levenberg-Marquardt nonlinear optimization method [12],[13],[14] can be used to update the centroid location of the ellipse. Taking the first-order Taylor expansion of the vector of functions $\mathbf{f}$ in the neighborhood of μ and neglecting the second-order term, it becomes:

$$\mathbf{f}(\mu+\delta\mu) \approx \mathbf{f}(\mu) + \mathbf{J_f}(\mu)\,\delta\mu \quad (8)$$

where $\mathbf{J_f}(\mu)$ is the Jacobian of the vector of functions $\mathbf{f}(\mu)$ and $\overrightarrow{\nabla} f_i(\mu)$ is the gradient of the function $f_i(\mu)$ at point μ:

$$\mathbf{J_f}(\mu) = \begin{pmatrix} \overrightarrow{\nabla} f_1(\mu) & \cdots & \overrightarrow{\nabla} f_N(\mu) \end{pmatrix}^T$$

Substituting (8) into (7) further yields the following approximation:

$$J(\mu + \delta\mu) = \|\mathbf{f}(\mu + \delta\,\mu)\|^2 \approx \|\mathbf{f}(\mu) + \mathbf{J}_\mathbf{f}(\mu)\,\delta\mu\|^2 \qquad (9)$$

It is noticed that Levenberg-Marquardt optimization method introduces a mixing term combining the steepest descent and the quadratic approximation with a weight factor η. When setting to zero the derivative of (9), we can find the best $\delta\mu$ by solving the following equation:

$$\left[\mathbf{J}_\mathbf{f}^T(\mu)\,\mathbf{J}_\mathbf{f}(\mu) + \eta\,\mathbf{I}\right]\delta\mu = -\mathbf{J}_\mathbf{f}^T(\mu)\,\mathbf{f}(\mu) \qquad (10)$$

where $\mathbf{I}$ is an identity matrix (of size 2×2 in our case).

(3) **Shape**: Similar to the ways the scale and centroid location of an ellipse are estimated, we can deform the shape of an ellipse in a way such that the following cost function is minimized:

$$J\left(\Sigma^{-1}\right) = \sum_{i=1}^{N}\left\|(\mathbf{x}_i - \mu)^T\ \Sigma^{-1}\ (\mathbf{x}_i - \mu) - c^2\right\|^2 \qquad (11)$$

As the covariance matrix Σ is symmetric, so is Σ^{-1}, i.e. $\Sigma^{-1} = \begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_2 & \gamma_3\end{pmatrix}$. Substituting this expression into (11) and setting its derivative to zero, the best shape update can be solved as:

$$\Gamma^* = \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^*\end{pmatrix} = \left(\Phi\Phi^T\right)^{-1}\Phi\vec{\mathbf{I}} \qquad (12)$$

where $\Phi = \begin{pmatrix} a_1^2 & a_2^2 & \cdots & a_N^2\\ 2a_1b_1 & 2a_2b_2 & \cdots & 2a_Nb_N\\ b_1^2 & b_2^2 & \cdots & b_N^2\end{pmatrix}$; a_i and b_i are the two entries of $\mathbf{x}_i - \mu$, i.e. $\begin{pmatrix}a_i & b_i\end{pmatrix}^T = \mathbf{x}_i - \mu$; $\vec{\mathbf{I}}$ is a vector consisting of N entries of 1.

Overall, this method keeps updating the scale, centroid, and shape of an ellipse and is an analogy to the deformable shape model that allows the active contour to adapt itself to the desired position. In this case, the image force field is determined by sample data while the internal energy is derived from the ellipse model. An algorithmic description of the proposed approach is summarized in Algorithm 1.

III. EXPERIMENTS AND RESULTS

To demonstrate our method's resistance to high curvature bias, we design a set of controlled experiments where data points are sampled from conic segments with low curvatures (cf. Fig. 2 (a)∼(d)). For a complete discussion on the effect of diverse data distributions on fitting performance, we also test on those fitting cases where data points are distributed around high curvature segments (cf. Fig. 2 (e)∼(h)). Since data is inevitably degraded by noise in reality, we introduce to uncorrupted data an additive standard Gaussian noise with η as its multiplicative weight. In addition, we compare the resulting fitting performances with those of *B2AC*, a representative method using the polynomial representation proposed in [7]. When $\eta = 0.05, 0.1, 0.15, \text{and}, 0.2$, fitting results from both methods are presented in Fig. 2, along with the underlying ellipses from which uncorrupted data points are generated.

Algorithm 1 The ellipse fitting algorithm using a bivariate normal distribution model.

Require: Given a dataset χ consisting of N pairs of data coordinates: $\chi = \{\mathbf{x}_i = \begin{pmatrix}x_{i_1} & x_{i_2}\end{pmatrix}^T, i = 1, 2, \ldots, N\}$

- Estimate the initial mean vector and the covariance matrix with the unbiased Maximum Likelihood Estimator (MLE): $\mu(0) \leftarrow \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i$; $\Sigma(0) \leftarrow \frac{1}{N-1}\sum_{i=1}^{N}(\mathbf{x}_i - \mu)(\mathbf{x}_i - \mu)^T$
- Initialize the step number $n \leftarrow 0$, and the upper limit for the number of iterations τ_N^*
- **For** $n = 0, 1, \cdots, \tau_N^* - 1$ **do**
  - Compute $\Sigma^{-1}(n)$; Compute $c^*(n+1)$ using (6).
  - Update the ellipse location shift $\delta\mu^*(n)$ by solving (10); $\mu^*(n+1) \leftarrow \mu^*(n) + \delta\mu^*(n)$.
  - For $i = 1, 2, \cdots, N$: compute $\begin{pmatrix}a_i & b_i\end{pmatrix}^T = \mathbf{x}_i - \mu$ and construct matrix Φ.
  - Compute $\Gamma^*(n+1)$ using (12).

  end for
- Compute $\Sigma^* \leftarrow \frac{1}{\gamma_1(n)\gamma_3(n) - (\gamma_2(n))^2}\begin{pmatrix}\gamma_3(n) & -\gamma_2(n)\\ -\gamma_2(n) & \gamma_1(n)\end{pmatrix}$.
- Diagonalize matrix Σ^*: $\Sigma^* = \mathbf{U}^*\,\Lambda^*\,\mathbf{U}^{*T}$.
- Compute the estimated ellipse points with:

$$\widehat{\chi} = \mathbf{U}^*\left(c^*(n)\,(\Lambda^*)^{\frac{1}{2}}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\right) + \mu^*(n) \text{ where } \theta \in [0, 2\pi)$$

Output: $\chi^\dagger = \{\widehat{\mathbf{x}}_j \mid \widehat{\mathbf{x}}_j \in \widehat{\chi}, j = 1, 2, \cdots, M\}$

It is noticed in Fig. 2 (a)∼(d) that no matter how strong the noise level is, the fitting results from our proposed method are more consistent to the underlying true ellipses and present better stability as compared to those of *B2AC*. Unlike *B2AC*, our method is much less subject to the high curvature bias that results in the increasing deviation of fitting results from high curvature sections as the degree of noise gets larger. When noisy data is sampled from conic segments with high curvature in Fig. 2 (e)∼(h), our new method and *B2AC* become to have similar fitting behaviors.

To evaluate the fitting results in a quantitative manner, a

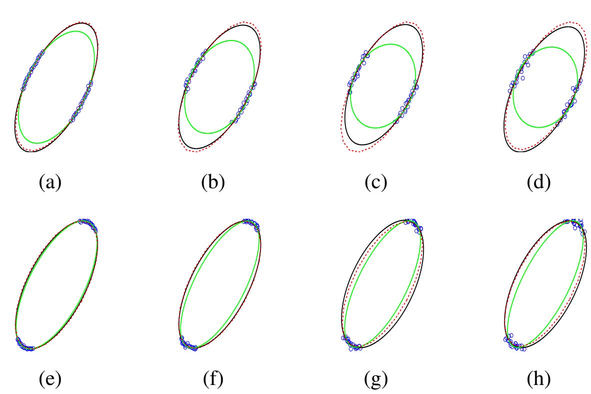

Fig. 2. When data (blue circle) close to low and high curvature sections of an ellipse (red dash) is coupled with additive Gaussian noise with $\eta = 0.05, 0.1, 0.15, \text{and}, 0.2$, fitting results produced by our method (black solid) and *B2AC* (green solid) are compared in (a)∼(d), and (e)∼(h), respectively.

TABLE I

COMPARISONS OF ζ'S FOR EXPERIMENTS IN FIG. 2 ARE PRESENTED.

	Data from low curvature		Data from high curvature	
Noise Strength	B2AC	Our Method	B2AC	Our Method
$\eta = 0.05$	0.1038	**0.0892**	0.1046	**0.1035**
$\eta = 0.10$	0.5430	**0.4118**	0.3276	**0.3027**
$\eta = 0.15$	1.0302	**0.7275**	0.7111	**0.7013**
$\eta = 0.20$	1.9022	**1.2667**	1.6231	**1.5484**

fitting error metric named as the Root of Sum of Squared sample-to-ellipse Orthogonal Distance (RSSOD) is defined as: $\zeta = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\|\mathbf{x}_i - \widehat{\mathbf{x}}_i\|^2}$, where $\widehat{\mathbf{x}}_i$ is the nearest point on the fitted ellipse to the data point $\mathbf{x}_i$ and N is the total number of fitting points to fit an ellipse with. The resulting fitting errors associated with experiments in Fig. 2 are presented in Table I for both methods.

To validate the efficacy of the proposed fitting algorithm for practical applications, we apply it to a data set consisting of 69 Optical Coherent Tomography (OCT) images of cross-sections of coronary artery stents with resolution of 500×500 in pixels. As a coronary artery stent is an expandable scaffold placed inside the lumina of a narrowed coronary artery for maintaining sufficient flow of heart-sustaining blood, its deformed shape, originally being circular, can be adequately represented by an ellipse under the pressure from blood vessels. Due to the nature of this imaging modality, the resulting images are often considerably degraded with noise and artifacts so that only few joint segments of stent profiles are captured. As a result, fitting an ellipse to detected stent segments can help characterize the original stent shapes and strut positions, and facilitates cardiologists visually assessing the appropriateness of the stent placement in the coronary artery after an implantation surgery. Additionally, it can also help cardiologists detect those cases where coronary arteries with implanted stents become narrow again, due to the fibrotic tissues built up on stent surfaces. Some typical fitting results are shown in Fig. 3 where yellow circles represent centroids of stent segments longer than 5 pixels detected by a steerable filter for finding local ridges [15].

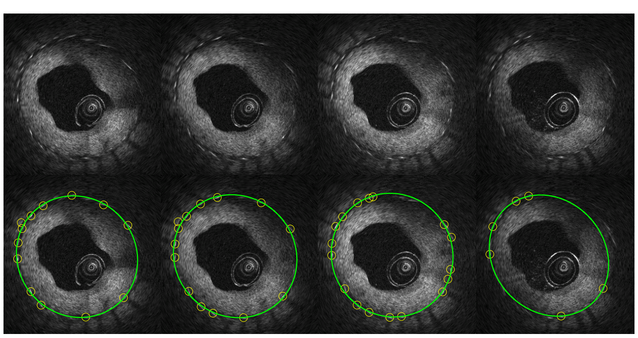

Fig. 3. A set of stent fitting results are presented. Top row: original OCT images of stents in coronary arteries; Bottom row: recovered stent profiles (green lines) from centroids of detected stent segments (yellow circles) by our ellipse fitting method.

The mean and standard deviation of the resulting ζ's associated with the 69 testing images are 3.15 and 1.53. Additionally, we also calculate the ensemble ζ of all the 69 images as a whole, i.e. N being equal to the total number of fitting points from these 69 images. The resulting aggregated ζ is 3.65. Given the fact that all these stent OCT images are of size 500×500 in pixels and of pixel resolution $10 \sim 20$ in microns, the computed ζ values are considered adequately small for satisfying the accuracy requirement of this application.

IV. CONCLUSIONS

In this paper, we formulate the optimal conic fitting problem with the use of a deformable bivariate normal distribution model. In our experiments, the proposed method presents both better fitting stability and higher resistance to the high curvature bias problem than a classical method following the routine second order polynomial representation. As a result, the new fitting method produces better fitting accuracies. Furthermore, we validate the practical efficacy of this method by successfully applying it to identifying profiles of coronary artery stents in a large set of OCT images with satisfying performances.

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