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On the Ill-conditioned Nature of the Intracardiac Inverse Problem

Jason H. T. Bates, *Senior Member*, *IEEE*, and Peter S. Spector

Abstract— **Multi-electrode catheters can be placed transvenously and positioned on the atrial endocardial surface in order to sample the chaotic electrical activity taking place during atrial fibrillation. We consider here the possibility of placing an array of electrodes over a relatively small, and hence roughly planar, region of the atrial surface in order to examine local activity patterns. This provides a spatially coarse but temporally fine sampling of electrical activity that can be expressed at each point in time as the convolution of the true electrical excitation of the tissue with a hyperbolic point spread function. We demonstrate the deconvolution of sampled signals using a polynomial approximation of the true electrical activity. When the deconvolution is unconstrained the inverse problem is poorly conditioned, showing that a high spatial sampling rate is required for accurate reconstructions of atrial activity in the vicinity of the electrode array. We discuss ways in which the conditioning of the problem might be improved through the application of constraints on the solution.**

I. Introduction

THE goal of intra-cardiac mapping is to determine the 3-dimensional activation sequence of cardiac tissue from recordings of electrical potential vs. time made from a small number of electrodes placed at various locations over the heart muscle. Given the complex and dynamic morphology of cardiac excitation during atrial fibrillation, achieving a detailed activation map would require fine electrode spacing. Currently, this is done clinically by using roving bipolar electrodes to localize activation to small regions of tissue near the electrodes. The mapping electrodes are moved throughout the chamber to sequentially sample endocardial electric potential leading to a map of "local activation time" relative to some reference point. Sequential mapping is based on the assumption that the activation pattern is stable throughout the procedure, a condition not satisfied during atrial fibrillation. It would therefore be extremely useful to be able to use multiple simultaneous recordings from unipolar electrodes to arrive at an activation map immediately.

Simultaneous recording from multiple intra-cardiac electrodes has been applied extensively within the ventricles using spherical arrays of electrodes. Solving the inverse problem with respect to these electrodes, based on an independent assessment of ventricular geometry, has yielded course assessments of ventricular activation patterns(1-4). The same procedure has also been applied with the atria (5). However, simultaneously mapping the entire internal atrial surface with the resolution necessary to resolve atrial fibrillation would require a huge number of intra-cardiac electrodes, and it may be some time before we have the technology to achieve this. Nevertheless, we can, at the present time, realistically contemplate using arrays of closely-spaced electrodes placed over small regions of the atria in order to obtain a detailed map of local activity. This could be useful in trying to understand the nature of excitation foci or regions where electrical activity appears to be particularly chaotic. Here we formulate the problem of resolving local activity from a region of tissue small enough to be considered planar, and examine some of the practical limitations faced in solving the inverse problem.

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J. H. T. Bates is with the Division of Pulmonary/Critical Care Medicine, Department of Medicine, University of Vermont and Fletcher Allen Health Care, Burlington, VT 05405 USA (phone: 802-656-8912; fax: 802-656-8926; e-mail: jason.h.bates@uvm.edu).

P. S. Spector is with the Division of Cardiology, University of Vermont and Fletcher Allen Health Care, Burlington, VT 05405 USA. (e-mail: peter.spector@lmed.uvm.edu).

II. Unipolar Electrode Recording

During the cardiac cycle, electrical activity spreads across the heart. The voltage (V) outside each cell is dynamic; the cardiac action potential is a cyclic variation of transmembrane voltage that occurs with each excitation. Using the monodomain approximation (6), the current density (I) associated with this flow of ions gives rise to an electric potential Φ at a distance r from the current source of

$$\Phi(r) = A\frac{I}{r} \tag{1}$$

where A embodies the electrical properties of the medium between the current source and the electrode, and is a constant if the electrical properties of the medium (i.e. blood) between the cell surface and the measurement point are uniform and isotropic (7, 8).

We consider an idealized situation in which the atrial tissue can be considered to be two-dimensional. This applies when we are mapping over a region of tissue of uniform thickness that is small enough in extent to be essentially a flat plane, such as might pertain when an array of electrodes is placed over a relatively small fraction of the entire inner atrial surface. When an electrode is placed at a height h above a sheet of activated atrial tissue (h is small compared to the dimensions of the tissue sheet) then the Φ recorded at the electrode at time t consists of a contribution from every cell in the sheet weighted inversely by the linear distance of the cell to the electrode (provided the electrodes are bathed in a well-mixed medium of uniform conductivity that connects them electrically to the heart muscle). That is,

$$\Phi(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{I(z,w,t)}{\sqrt{(x-z)^2+(y-w)^2+h^2}}\,dzdw \tag{2}$$

where z and w are dummy spatial variables of integration. $\Phi(x, y, t)$ thus consists of the current source distribution $I(z, w, t)$ convolved with the hyperbolic point-spread function $[(x-z)^2 + (y-w)^2 + h^2]^{-1/2}$ (9).

III. A Polynomial Approach to the Inverse Problem

A. The Electrode Potential as a Convolution

The inverse problem consists of estimating $I(z, w, t)$ from $\Phi(x, y, t)$. In reality, Φ is sampled with only a small number of electrodes. Φ is therefore available with high temporal resolution but poor spatial resolution. The goal of the inverse problem is to estimate $I(x, y, t)$ from this small number of recordings of Φ made simultaneously at discrete sites over the atrial surface.

For ease of development, we will consider this problem in one dimension. Equation 2 becomes

$$\Phi(x,t) = \int_{-\infty}^{\infty} \frac{I(z,t)}{\sqrt{(x-z)^2 + h^2}} dz = I(x,t) * \frac{1}{\sqrt{x^2 + h^2}} \tag{3}$$

where * denotes the operation of convolution.

B. Estimating Current Density by Deconvolution

Estimating $I(x,t)$ from $\Phi(x,t)$ is a deconvolution problem that, in principle, should be amenable to inversion via Fourier transformation in the usual way. However, the spacing between the electrodes may not be uniform, which means that interpolation of $\Phi(x)$ between the samples to sites of equal spacing is necessary before Fourier-based deconvolution can be attempted. Also, the electrode array will likely not cover the entire atrial surface, but will rather be localized to a particular region of interest. This will result in sampling of a spatially truncated version of the global cardiac electrical activity. Deconvolving out the effects of electrode height on such a sample using Fourier methods will necessitate dealing with edge effects to avoid leakage. Therefore, in an attempt to avoid having to deal with above practical issues related to the use of the Fourier transform, we have explored the use of a polynomial method for inverting Eq. 3 to estimate $x(t)$.

Approximating I at any instant in time by a polynomial function of x gives

$$\hat{I}(x,t) = \sum_{j=0}^{N} a_j(t) x^j \tag{4}$$

where $\hat{I}$ is the approximation to I. The potential recorded at the ith electrode is then approximately

$$\hat{\Phi}_i(x,t) = \sum_{j=0}^{N} a_j(t) \left[\int_{-\infty}^{\infty} \frac{z^j}{\sqrt{(x-z)^2 + h^2}} dz \right] \tag{5}$$

where $\hat{\Phi}$ is the approximation to Φ.If we define

$$\Omega_{i,j}(t) = \int_{=\infty}^{\infty} \frac{z^j}{\sqrt{(x_i - z)^2 + h^2}} dz, \tag{6}$$

then

$$\hat{\Phi}_i(t) = \sum_{j=0}^{N} \Omega_{i,j}(t) a_j(t). \tag{7}$$

If the number of electrodes is greater than $N + 1$ then the a_j can be estimated by multiple linear regression. Otherwise, finding the a_j is simply a matter of matrix inversion.

IV. Numerical Examples

Suppose that $I(x)$ is a smooth, low-order function of x, such as that shown defined over the range $x \in [0, 8]$ and shown as the solid line in Fig. 1. If $I(x)$ were measured at any point along over this range with an electrode of height $h = 1$, it would appear via Eq. 3 as the widened function $\Phi(x)$ shown as the dashed line in Fig. 1. Fitting Eq. 7 to $\Phi(x)$ for $N = 4$ produces the fitted function shown as the closed circles in Fig. 1. Finally, evaluating the polynomial expression for $I(x)$ (Eq. 4) gives the open circles shown in Fig. 1.

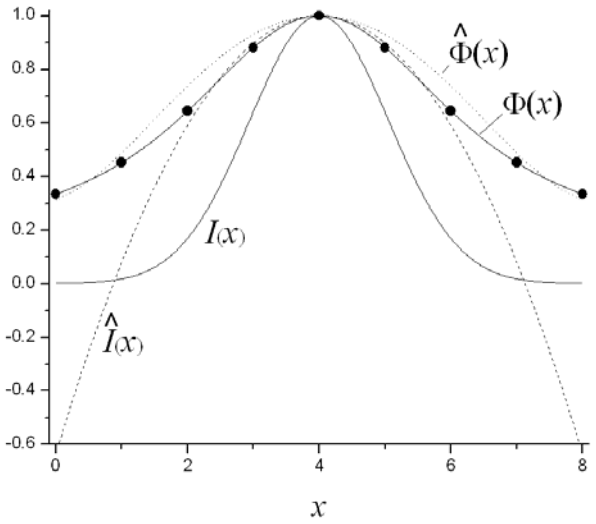

Fig. 1. A Gaussian $I(x)$ is shown as the solid line. The potentials $\Phi(x)$ that would be measured by 8 electrodes at a height of 1 above the tissue are shown as the filled circles. The estimate of $I(x)$ provided by Eq. 4 is shown as the dotted line. The estimate of $\Phi(x)$ provided by Eq. 5 is shown as the dashed line. All curves have been normalized to have a maximum of 1.0. Length (x) is in arbitrary units.

In this particular example, the polynomial deconvolution scheme embodied in Eqs. 4 through 7 returns an estimate of $I(x)$ that is closer to the true signal than is the measured $\Phi(x)$. One the other hand, the estimated and true $I(x)$ signals differ substantially from each other in some places, particularly near the edges of the [0, 8] range of x. Despite these differences, however, the true and fitted $\Phi(x)$ signals are very close to each other at the 9 electrode locations. This illustrates the ill-conditioned nature of the intra-cardiac deconvolution problem; there is a wide range of possible inverse solutions for $I(x)$ that can explain a given set of discrete measurements of $\Phi(x)$, which places a practical limit on how sparsely the electrical activity can be sampled over the internal surface of the atrium before information about the electrical activity between the samples cannot be

uniquely recovered.

The polynomial deconvolution approach runs into further difficulties, however, when the spatial frequency of $I(x)$ is too great to be accurately described by a low-order polynomial, as exemplified in Fig. 2 where $I(x)$ is still a smooth function of x, but has more degrees of freedom than in Fig. 1. Now the fit to $\Phi(x)$ provided by Eq. 7 is rather poor, and the recovered $I(x)$ provided by Eq. 4 is further away from the true $I(x)$ than is $\Phi(x)$ itself. In this case, one would do better not to have attempted deconvolution at all.

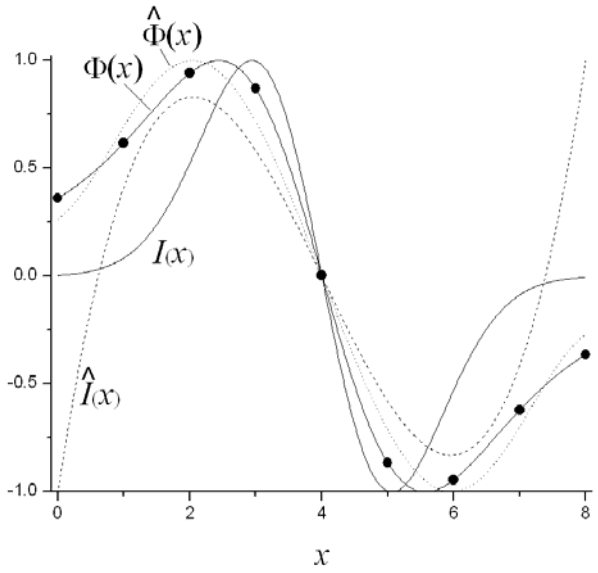

Fig. 2. The polynomial deconvolution method has difficulties with a more tortuous $I(x)$ signal (compare to Fig. 1).

V. Conclusions

We have outlined a simplified 1-dimensional version of a problem that, strictly speaking, requires solving Maxwell's equations in an appropriate 3-dimensional volume with a geometry corresponding to that of the atrium and the electrode array. A significant amount of work has already been done on this problem, particularly with regard to mapping patterns of ventricular activation (2, 4, 10, 11), so this is not an unexplored area. However, for small electrode arrays localized to specific areas of the atrial tissue, a simplified analysis of the type outlined above may be appropriate and useful. This would allow a more precise analysis of the local electrical activity which may facilitate characterization of atrial electrical activity during atrial fibrillation. On the other hand, the deconvolution problem is clearly an ill-conditioned one, so accurate reconstruction of spatial current density distributions over the atrial surface will require a sufficiently high spatial frequency of sampling, and the incorporation of as many constraints on the nature of the solution as possible. Such constraints could include 1) limiting the excursion of the solution between fixed limits, 2) forcing the solution toward finite asymptotes at large distances from the electrode array, and 3) limiting the variation between solutions at adjacent time points.

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