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The Effects of Non-Linearities on Shear Stress in Periodic Flow in Axi-Symmetric Vessels

Chang Sub Park and Stephen John Payne

Abstract— **In this paper, a power series and a Fourier series approach is used to solve the governing equations of motion in an elastic axi-symmetric vessel, assuming that blood is an incompressible Newtonian fluid. The time averaged flow has shown to be greater than the steady state flow leading to a larger wall shear stress. Oscillations can also be observed, which is not present in the steady state solution. This is due to the nonlinear momentum terms causing interaction between the harmonics.**

I. INTRODUCTION

Mathematical modelling of physiological flow is a useful tool to understand and predict physiological effects in the vascular system. Wall shear stress in particular has been related to vascular diseases such as atherosclerosis [1], which can be fatal if left untreated. The modelling of the vascular flow is complex due to the unsteady non-linearity of the governing equation and its interaction with the vessel wall. Little work has been done to analyse the non-linear behaviour of the flow.

A novel technique is thus proposed here which applies a non-linear two dimensional, axial and radial, algorithm to simulate blood flow in a single axi-symmetric elastic vessel with no curvature. A power series over the radius and a Fourier series over time is used to represent the velocity whilst the radius of the vessel is represented using a Fourier series. These are substituted in to the governing equations of motion converting the partial differential equations, PDEs, in to a series of coupled ordinary differential equations, ODEs, which significantly reduces the computational cost. A similar algorithm to the one used here has been explained in more detail in [2]. Wall shear stress depends on the velocity profile and the variation of it along the axial length will be examined. The effects of the stiffness and Reynolds number on the oscillations will also be examined here.

II. THEORY

A. Governing equations of motion

The governing equations of motion for an incompressible fluid can be obtained from the Navier-Stokes equations. For an axi-symmetric vessel with no curvature, this can be expressed in polar form (x, r). Taking orders of magnitude [3], the governing equations can be expressed as:

$$\frac{\partial U_x}{\partial t} + U_r \frac{\partial U_x}{\partial r} + U_x \frac{\partial U_x}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu \left(\frac{\partial^2 U_x}{\partial r^2} + \frac{1}{r}\frac{\partial U_x}{\partial r} \right), \quad (1)$$

$$\frac{\partial p}{\partial r} = 0, \quad (2)$$

and the continuity equation as:

$$\frac{\partial U_x}{\partial x} + \frac{1}{r}\frac{\partial (r U_r)}{\partial r} = 0, \quad (3)$$

where U_x and U_r are the respective axial and radial velocity components, ρ is the density of the fluid, p the pressure and ν the kinematic viscosity.

A relationship between the pressure and the radius of the vessel is required to solve the equations. The most common form is a simple power-law relationship [4], of the form:

$$p - p_E = G_0 \left(\frac{R}{R_E} - 1 \right), \quad (4)$$

where p_E is the pressure of the vessel at the equilibrium state, R is the radius of the vessel, R_E is the radius of the vessel at the equilibrium state and G_0 the wall stiffness. Equation (4) thus assumes a linear relationship between the pressure and the radius and this is used here as a first step for simplicity.

B. Trial solution

To solve the governing equations, a power series approach over radius and a Fourier series over time is proposed. The axial velocity is thus of the form:

$$U_x(x, r, t) = \sum_{m=0}^{M} \sum_{n=-N}^{N} U_{m,n}(x) \left\{ \frac{r}{R(x,t)} \right\}^m e^{\imath n \omega t}, \quad (5)$$

where the $U_{m,n}$ values are the coefficients of the series, ω is the angular frequency and R is the radius of the vessel which, using the same approach as above, can be written as:

$$R(x, t) = \sum_{q=-Q}^{Q} R_q(x)\, e^{\imath q \omega t}, \quad (6)$$

where R_q are the coefficients of the radius series. The power series over radius is in this form, r/R, so that it is possible to set the boundary conditions at the walls. The Fourier series assumes that the flow is periodic. This is very relevant to the cardiovascular system due to its strong periodicity. The trial solution of the axial velocity (5), provides a complete solution when M,N,$Q \to \infty$, and approximate solutions can

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C.S. Park is a doctorate student at the Department of Engineering Science, University of Oxford, Oxford, UK chang.park@eng.ox.ac.uk

S.J. Payne is with the Department of Engineering Science, University of Oxford, Oxford, UK stephen.payne@eng.ox.ac.uk

$$St'\imath\sum_{m=0}^{M}\sum_{n=-N}^{N}u_{2m,n}y^{2m}e^{\imath nt'}\left(n\sum_{q=-Q}^{Q}Y_qe^{\imath qt'}-2m\sum_{q=-Q}^{Q}qY_qe^{\imath qt'}\right)+\sum_{m=0}^{M}\sum_{n=-N}^{N}\frac{y^{2m}}{2\left(m+1\right)}e^{\imath nt'}$$

$$\cdot\left(2mu_{2m,n}\sum_{q=-Q}^{Q}\frac{dY_q}{dz}e^{\imath qt'}-\frac{du_{2m,n}}{dz}\sum_{q=-Q}^{Q}Y_qe^{\imath qt'}\right)\sum_{m=0}^{M}\sum_{n=-N}^{N}2mu_{2m,n}y^{2m}e^{\imath nt'}$$

$$+\sum_{m=0}^{M}\sum_{n=-N}^{N}u_{2m,n}y^{2m}e^{\imath nt'}\sum_{m=0}^{M}\sum_{n=-N}^{N}y^{2m}e^{\imath nt'}\left(\frac{du_{2m,n}}{dz}\sum_{q=-Q}^{Q}Y_qe^{\imath qt'}-2mu_{2m,n}\sum_{q=-Q}^{Q}\frac{dY_q}{dz}e^{\imath qt'}\right)$$

$$+\kappa\sum_{q=-Q}^{Q}\frac{dY_q}{dz}e^{\imath qt'}\sum_{q=-Q}^{Q}Y_qe^{\imath qt'}=\frac{4}{\varepsilon ReY_0}\sum_{m=0}^{M}\sum_{n=-N}^{N}(m+1)^2u_{2(m+1),n}\,y^{2m}e^{\imath nt'}\sum_{p=0}^{\infty}(-1)^p\left(\sum_{\substack{q=-Q\\q\neq0}}^{Q}\frac{Y_q}{Y_0}e^{\imath qt'}\right)^p. \quad (7)$$

be obtained for finite values of M, N and Q. In practice, we set $N=Q$ since the highest harmonics of interest in both cases are likely to be the same.

By substituting the trial solution to the continuity equation (3), the radial velocity can be directly derived. Both solutions can be substituted to the governing equations of motion (1). Re-arranging and non-dimensionalising the variables such that $y=r/R$, $z=x/L$, $t'=\omega t$, $u_{m,n}=U_{m,n}/U$ and $Y=R/R_E$ (L is initial vessel length and U is the characteristic velocity), the momentum equation (1) can be written as (7), where $St'=\omega L/U$ is the scaled Strouhal number, $\kappa=G_0/\rho U^2$, $\varepsilon=R_E/L$ and $Re=UR_E/\nu$ is the Reynolds number. Equation (7) is the final equation. Note that there are only three non-dimensional quantities, εRe, St' and κ, required to formulate the complete solution. It is possible to express (7) in matrix form as:

$$\mathbf{A}\frac{d}{dz}\left(\mathbf{u}\right)+\mathbf{B}\frac{d}{dz}\left(\mathbf{Y}\right)+\mathbf{c}=0, \quad (8)$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices and $\mathbf{c}$ is a vector. The matrices and vector are not expressed in full due to space limits.

To solve numerically, the boundary conditions need to be considered. At the vessel walls a no-slip condition is assumed here, although any suitable boundary conditions could be applied as desired. The non-dimensional axial velocity, u, at the vessel wall is equal to zero, therefore:

$$u(z,y=1,t')=\sum_{m=0}^{M+1}u_{2m,n}=0. \quad (9)$$

The radial velocity is equal to the rate of change of the radius with time, which in non-dimensional terms is given as:

$$v\left(z,y=1,t'\right)=\frac{St'}{Y}\frac{\partial Y}{\partial t'}, \quad (10)$$

where v is the non-dimensional radial velocity. This leads to the following equation:

$$\sum_{m=0}^{M+1}\frac{1}{2\left(m+1\right)}\left(2mu_{2m,n-q}\frac{dY_q}{dz}-\frac{du_{2m,n-q}}{dz}Y_q\right)=St'\imath nY_n, \quad (11)$$

which must be satisfied at each harmonic, from $-N$ to N. The number of unknowns is matched by the number of equations, and thus it is possible to solve this problem.

III. RESULTS

The aorta was considered here as the assumption of a Newtonian fluid makes it the most relevant. Considering the vessel dimensions and taking the average base velocity to be 0.5m s^{-1}, the wall stiffness to be 50kPa [5] and the fundamental frequency to be 1.2Hz results in the following non-dimensional quantities: $\varepsilon Re=156$, $\kappa=190$ and $St'=12\pi/25$. These initial values can be varied to observe their effects on the flow field.

A. Steady state behaviour

Considering a quadratic velocity profile at the inlet, $u=u_{0,0}\left(1-y^2\right)$ will result in a Poiseuille velocity profile regardless of the number of terms considered in the power series. The results are not shown here due to the limited space available. There is an increase in the velocity and a gradual decrease in the radius along the vessel. This agrees with the fact that in steady state, the flow rate is constant along the axial length. There is a decrease in the pressure along the vessel length due to the friction. This subsequently reduces the vessel radius due to its linear relationship with the pressure (4). The velocity therefore has to increase.

An expression for the radial velocity can be obtained from (3). The radial velocity is towards the centreline when the vessel contracts. However, this is not very significant since the radial velocity is so small compared to the axial velocity. This is due to the non-dimensional term εRe which is much larger than 1.

B. Dynamic behaviour (first harmonic)

A quadratic velocity profile, $u=u_{0,0}\left(1-y^2\right)$, was set at the inlet whilst the vessel wall was oscillated at the fundamental frequency with an amplitude of 0.05, such that $Y=Y_0\left\{1+0.025\left(e^{\imath t'}+e^{-\imath t'}\right)\right\}$. Fig. (1) shows the ratio of the time averaged to steady state velocity coefficient and how this changes with increasing terms in the power series.

A plot for 100 non-dimensional lengths is shown here to observe and clearly understand the results.

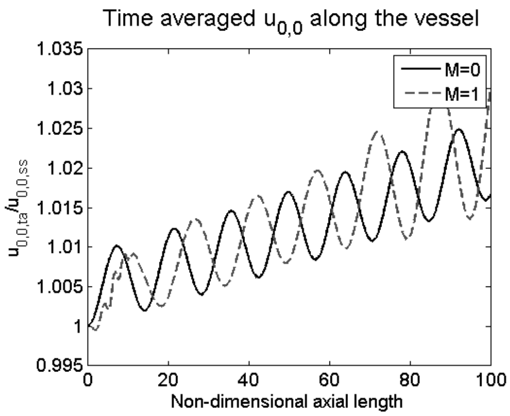

Fig. 1. Ratio of time averaged to steady state velocity coefficient for increasing M

Clearly there is a significant difference between the time averaged and the steady state velocity coefficients. Oscillations in the velocity coefficient can be observed in the time averaged case whilst the steady state, although not shown here, is almost linear. The oscillation is due to the non-linearity of the momentum terms in the governing equations of motion, which causes the interaction between the harmonics. Hence, in this case, it is the first harmonic terms which causes the observed crests and troughs. The physiological meaning of these are not known and requires further work. The difference increases with increasing the order of the power series considered.

The difference in the velocity coefficients between the time averaged and the steady state also implies that there will be a difference in the wall shear stress. The complete solution, including the first harmonic terms predicts flow reversal since the first harmonic terms are larger, in magnitude, than the zeroth harmonic terms. The results are not shown here due to the space limits. The number of terms required in the power series for an accurate solution has been considered in [2].

Fig. (2) shows the ratio of the time averaged to steady state radius coefficient and how this changes with increasing terms in the power series. The decrease in radius results in an increase in velocity from continuity. The amplitude of the oscillations increases with increasing M for the velocity plot whilst it decreases with increasing M for the radius plot. The order of the power series has also an effect on the wavelength of the oscillations which in turn has an effect on the wave speed. The increase in the order of terms considered results in an increase in the wave speed.

The non-dimensional wall shear stress, τ_w, can be expressed as:

$$\tau_w = \left.\frac{\partial u}{\partial y}\right|_{y=1} = \sum_{m=1}^{M+1} \sum_{n=-N}^{N} 2mu_{2m,n}e^{\imath nt'}. \qquad (12)$$

Fig. (3) shows the ratio of the time averaged to steady state wall shear stress. The non-dimensional axial length has been

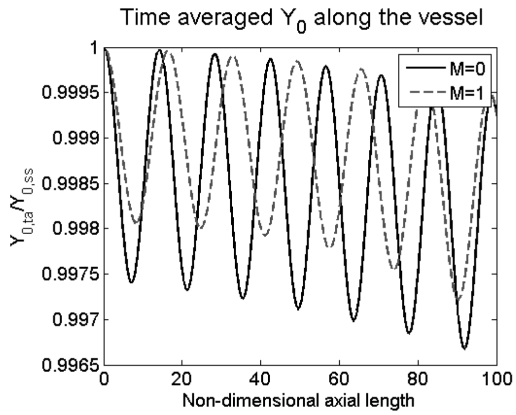

Fig. 2. Ratio of time averaged to steady state radius coefficient for increasing M

scaled so that the crests and troughs for both plots lie in phase. Once again, it is clear that there is a difference in the shear stress between the time averaged and the steady state. It is also dependent, as expected, with the order of the power series. The time averaged velocity profile is quadratic since the higher order terms are almost zero and thus not significant. However, it will have a significant effect on the wall shear stress due to the $2m$ coefficient, as shown in (12). Hence, as $M \to \infty$, the higher order terms in the power series have a greater effect on the wall shear stress.

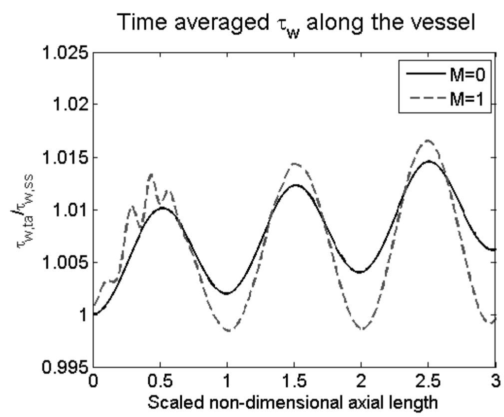

Fig. 3. Ratio of time averaged to steady state zeroth harmonic wall shear stress for increasing M

The complete solution including the first harmonic terms predicts oscillations of the wall shear stress with regions of reversed wall shear stress as expected. Once more, these results are not shown here due to the limited space available.

The oscillations of the plots will be more significant as the stiffness of the vessel wall is decreased as shown in fig. (4). Reducing the wall stiffness leads to an increase in the amplitude of the oscillations as well as an overall increase in the velocity coefficient. This is as expected since the stiffer the vessel wall, the smaller the oscillations in the vessel wall which leads to smaller oscillations in the velocity.

Fig. (5) shows the plot of the velocity coefficient for different Reynolds number. Considering $\varepsilon Re \to \infty$, no

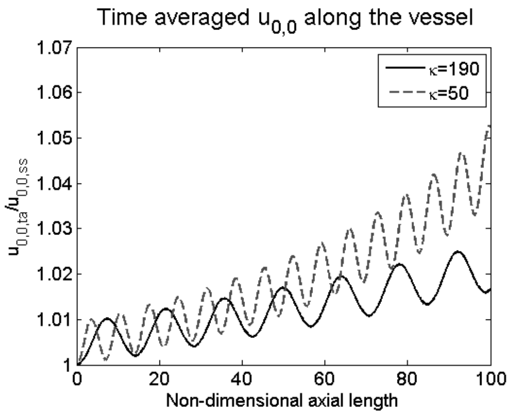

Fig. 4. Time averaged velocity coefficient for different wall stiffness

viscous effects, the solution would be for an inviscid flow. The results show that the εRe term has no effect on the amplitude of the oscillations as these are similar nor it has an effect on the wavelength of the oscillations. However, it does have an effect on the overall magnitude of the velocity coefficient along the vessel, this increasing as the viscous terms have a greater influence.

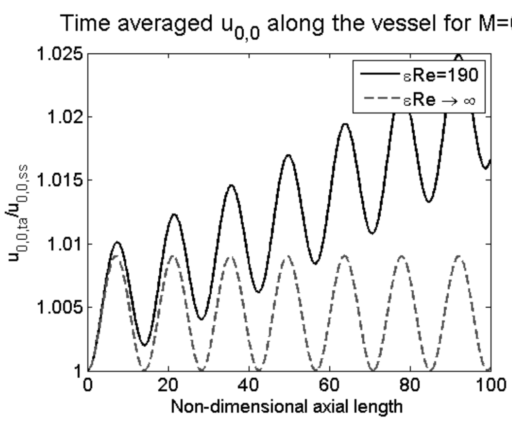

Fig. 5. Time averaged velocity coefficient for different Reynolds number

IV. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

A two dimensional algorithm to solve for the velocity, pressure and wall shear stress of elastic axi-symmetric vessels has been presented here. The governing PDEs are converted into a series of simultaneous ODEs which can reduce considerably the computational cost. An elastic vessel with a quadratic inlet velocity profile was considered for both steady and unsteady flow. The model predicts that the Poiseuille flow is a good approximation for fully developed steady flow. The unsteady flow predicts physiological effects observed from experiments. The equations presented in this paper show that there are only three non-dimensional parameters that govern the effects of the flow: St', κ and εRe. The order of the power series considered is important as it has an effect on the wave speed. The effects these parameters and the order considered have on the fluid motion and wall motion, as well as the validity of the preliminary results obtained, are described in more detail in [2].

There is a clear difference between the time averaged and the steady state solutions. This is due to the non-linear momentum terms which lead to the interaction of the harmonics. Spatial lag between the motion of the wall and the fluid motion can be observed whilst the fluid motion and the wall shear stress are in phase. The order of the power series has an effect on the wavelength of the oscillations which results in different wave speeds. The time averaged solution for both the velocity and wall shear stress is greater than the respective steady state solution. The wall stiffness has an effect on both the amplitude of the oscillations and the overall magnitude of the velocity coefficients whilst the Reynolds number has no effect on the oscillations.

B. Future Works

The difference in the time averaged and steady state wall shear stress needs to be studied in more detail due to its physiological importance. The effects the oscillations of the zeroth harmonic have physiologically requires further investigation. The number of terms considered in the power series, the harmonics considered in the Fourier series and their respective effects on the fluid motion and wall motion require further work. Comparison with experimental results would be required to further validate our model.

The Navier-Stokes equation can be coupled with the mass transport equation to obtain the concentration profile of a substance in the vascular network. Nitric oxide is believed to have an effect on vasodilation which in turn plays a role in vascular homeostasis. A linear pressure-radius relationship was considered here for simplicity. This can be modified by changing the power of the radius leading to a non-linear relationship. This would lead to two complex non-linear systems to interact with each other giving rise to more complex behaviour. It is also possible to vary the wall stiffness along the vessel and observe its effects on the velocity profile, giving better insights to aneurysms and atherosclerosis. The method used here is not restricted to just vascular vessels but could be used to solve for flow in general conduit vessels since the the mathematical algorithm presented gives general solutions for flow in a vessel.

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