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Nonlinear Solution for Radiation Boundary Condition of Heat Transfer Process in Human Eye

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Abstract— In this paper we propose a new method based on finite element method for solving radiation boundary condition of heat equation inside the human eye and other applications. Using this method, we can solve heat equation inside human eye without need to model radiation boundary condition to a robin boundary condition. Using finite element method we can obtain a nonlinear equation, and finally we use nonlinear algorithm to solve it. The human eye is modeled as a composition of several homogeneous regions. The Ritz method in the finite element method is used for solving heat differential equation. Applying the boundary conditions, the heat radiation condition and the robin condition on the cornea surface of the eye and on the outer part of sclera are used, respectively. Simulation results of solving nonlinear boundary condition show the accuracy of the proposed method.

I. INTRODUCTION

TEMPERATURE may cause considerable effects on the human eye. By developing an advanced model of heat flow in the eye, it will be possible to understand steady state and transient behaviours of temperature distribution in the eye before and after thermal events. A finite element method is applied to analyze the temperate distribution in a three-dimensional (3D) model of the human eye. The human eye is modeled as a composition of several homogeneous regions and each region has its own physical and geometry properties [1]. For solving heat equation in human eye we face to complicated boundary condition which is named radiation boundary condition and in most of models such as transient heat transfer [1] or steady-state heat transfer modeling inside the human eye with two-dimensional (2D) and 3D models [2], this boundary condition is simplified to a robin boundary condition. In this paper, we present mathematical solution for radiation boundary condition using Ritz method in finite element method and it can help us to solve heat equation inside the human eye without modeling radiation boundary to robin boundary condition. Using this method we obtain a nonlinear equation, and finally using nonlinear algorithms, we can solve our nonlinear equations. Simulation results show the accuracy of this algorithm.

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II. DESCRIPTION OF THE HUMAN EYE TISSUES

The eye is approximately a spherical organ, figure 1 shows the eye and different components of the eye. The protective outer layer of the eye is called the sclera and it maintains the shape of the eye. The front portion of the sclera, called the cornea, is transparent and allows light to enter the eye. The cornea is a powerful refracting surface, providing much of the eye's focusing power. The iris is the part of the eye that gives it color. It consists of muscular tissue that responds to surrounding light, making the pupil, or circular opening in the center of the iris, larger or smaller depending on the brightness of the light. Light entering the pupil falls onto the lens of the eye where it is altered before passing through the retina. The lens is a transparent, biconvex structure, encased in a thin transparent covering. The function of the lens is to refract and focus incoming light onto the retina for processing. The retina is the innermost layer in the eye. It converts images into electrical impulses that are sent along the optic nerve to the brain where the images are interpreted. The inside of the eyeball is divided by the lens into two fluid-filled sections. The larger section at the back of the eye is filled with a colorless gelatinous mass called the vitreous humor. The smaller section in the front contains a clear, water-like material called aqueous humor [1].

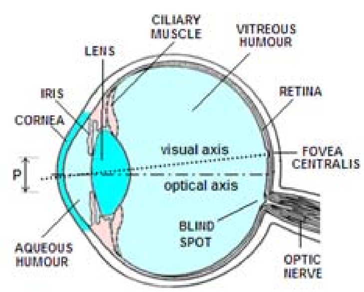

Fig 1. Human eye [3]

III. MODEL OF THE HUMAN EYE

A. Heat equation

For 2&3 dimensional model of human eye, some simplifications are adopted regarding geometry and modeling process such as the human eye is a solid structure consisting of a various tissues in contact with each other. Therefore, the eye is divided into seven regions. Furthermore, each region is assumed homogeneous. A number of heat transfer equations for living tissues have been developed. In this paper we use Pennes heat equation as follows [4]:

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$$\frac{\partial}{\partial x}\left(k_{i_x}\frac{\partial T_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{i_y}\frac{\partial T_i}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_{i_z}\frac{\partial T_i}{\partial z}\right)+Q_i$$
$$=\rho_i c_{i_p}\frac{dT_i}{dt}+\beta T_i \qquad i=1,2\ldots 7 \qquad (1)$$

Where k is the thermal conductivity, T is the temperature, Q is generalized source term ρ is the density, c is the specific heat, and β is a constant. The index i indicates a region of human eye, in particular the cornea, aqueous humour, iris, lens, vitreous, sclera and optic nerve. In human eye, no heat source is considered, Q=0 and it is assumed $\beta = 0$. The physical parameters for all regions are tabulated in table I. Therefore, the heat equation can be expressed:

$$\frac{\partial}{\partial x}\left(k_{i_x}\frac{\partial T_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{i_y}\frac{\partial T_i}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_{i_z}\frac{\partial T_i}{\partial z}\right)=\rho_i c_{i_p}\frac{dT_i}{dt} \qquad (2)$$

For steady-state problem the above equation is simplified to:

$$\frac{\partial}{\partial x}\left(k_{i_x}\frac{\partial T_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{i_y}\frac{\partial T_i}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_{i_z}\frac{\partial T_i}{\partial z}\right)=0 \qquad (3)$$

Table I. Thermo physical parameters of a human eye [5].

Region	$C[J/(KgK]$	$\rho[Kg/m^3]$	$k[W/(mK)]$
Cornea	4178	1050	0.58
Iris	3997	1000	1.0042
Aqueous humour	3997	1000	0.58
Lens	3000	1050	0.40
Sclera	4178	1000	1.0042
Vitreous	4178	1000	0.603
Optic nerve	3997	1000	1.0042

For the 2D model the equation (2) is simplified to:

$$\frac{\partial}{\partial x}\left(k_{i_x}\frac{\partial T_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{i_y}\frac{\partial T_i}{\partial y}\right)=\rho_i c_{i_p}\frac{dT_i}{dt} \qquad (4)$$

B. *Boundary conditions*

In problem for solving we have three kinds of boundary conditions which are named Dirishlet, Robin and Radiation. The Dirichlet boundary condition gives us the temperature values on the boundary surface S of the studied field like a space function constant or variable in time in following form:

$$T=g(x,y,z,t) \qquad (5)$$

The Robin boundary condition gives us the value of the density of heat flow rate through the S boundary surface of the studied field in following form:

$$\alpha(T-T_a)=\lambda_x\frac{\partial T}{\partial x}n_x+\lambda_y\frac{\partial T}{\partial y}n_y+\lambda_z\frac{\partial T}{\partial z}n_z \qquad (6)$$

Where α is the convective heat transfer coefficient from S to the fluid (or inversely), λ is the thermal conductivity and T_a is the fluid temperature. The radiation boundary condition is like to robin boundary condition, but in this boundary condition we can see the term T^4 and T_a^4 in this boundary condition in following form:

$$\alpha(T-T_a)+\beta\left(T^4-{T_a}^4\right)=\lambda_x\frac{\partial T}{\partial x}n_x+\lambda_y\frac{\partial T}{\partial y}n_y+\lambda_z\frac{\partial T}{\partial z}n_z \qquad (7)$$

The boundary conditions used for 2D and 3D model is similar to each other. Two kinds of boundary conditions are used. The first boundary condition which is named radiation boundary condition is applied on the corneal surface where three heat loss mechanisms take place. These losses are the heat transfer due to convection, radiation and the heat loss as a result of tear evaporation from the corneal surface. Mathematically, this may be written as [6]:

$$-k_1\frac{\partial T_1}{\partial n}=\alpha_a(T_1-T_a)+\varepsilon\sigma\left({T_1}^4-{T_a}^4\right)+E; \quad \text{on } S_1 \qquad (8)$$

Where k_1 is thermal conductivity of cornea, $\alpha_a = 10\,W/(m^2K)$ is heat transfer coefficient between cornea and environment, $T_a = 25°C$ is the temperature of surrounding environment, $\varepsilon = 0.975 W/m^3$ is the corneal emissivity, $\sigma = 5.67\times10^{-8} W/(m^2K^4)$ is the Stefan-Boltzmann constant and $E = 40 W/m^2$ is the loss of heat flux due to the evaporation of tears. The second boundary condition which is named robin boundary condition is applied on the scleroid surface. The human eye is considered to be embedded in a homogeneous surrounding anatomy, which is at body core temperature. Consequently, heat transfer from the surrounding environment to the eye may be described by a single heat transfer coefficient as follows [6]:

$$-k_2\frac{\partial T_2}{\partial n}=\alpha_{bl}(T_2-T_{bl}); \quad \text{on } S_2 \qquad (9)$$

Where k_2 is the thermal conductivity of sclera, $\alpha_{bl} = 65\,W/(m^2K)$ is the heat transfer coefficient between sclera and blood vessels and $T_b = 37°C$ is the temperature of blood. The boundary condition is shown in figure 2, the red region refers to the corneal surface and blue region refers to scleroid surface.

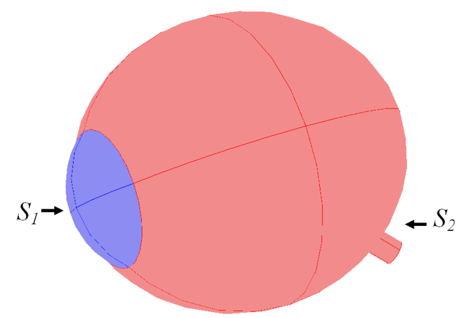

Fig2. Two boundary conditions on the human eye surface

IV. FINITE ELEMENT METHOD

In this paper, Ritz method in the finite element method is used to solve the heat equation in the human eye. The Ritz method also known as the Rayleigh-Ritz method is a variation method in which the boundary-value problem is formulated in terms of a variation expression, called functional. The minimum of this functional corresponds to the governing differential equation under the given boundary conditions. The approximate solution is then obtained by minimizing the functional with respect to variables that define a certain approximation to the solution [1, 6]. A typical boundary-value problem can be defined by a governing differential equation in Ω domail in below:

$$\mathcal{L}\phi=f \qquad (10)$$

In the above equation $\mathcal{L}$ is differential operator, f is excitation or forcing function and ϕ is unknown quantity. Using Ritz method, we can compose a variational expression which is named functional. The minimum of this functional corresponds to the governing differential equation under the given boundary conditions. The approximation solution is then obtained by minimization the functional. To illustrate procedure, the inner product, denoted by angular brackets is:

$$\langle \mathcal{L}\phi, \psi \rangle = \int \phi\psi^* d\Omega \tag{11}$$

In the above equation the operator $\mathcal{L}$ is self-adjoint and:

$$\langle \mathcal{L}\phi, \psi \rangle = \langle \psi, \mathcal{L}\phi \rangle \tag{12}$$

And:

$$\langle \mathcal{L}\phi, \phi \rangle = \begin{cases} > 0 & \phi > 0 \\ 0 & \phi = 0 \end{cases} \tag{13}$$

$\tilde{\phi}$ is a trial function which must be determined, therefore the functional for minimization is:

$$F(\tilde{\phi}) = \frac{1}{2}\langle \mathcal{L}\tilde{\phi}, \tilde{\phi} \rangle - \frac{1}{2}\langle \tilde{\phi}, f \rangle - \frac{1}{2}\langle f, \tilde{\phi} \rangle \tag{14}$$

Therefore, solving the equation is equivalent to minimize the above equation. By minimizing the above equation we obtain:

$$K\tilde{\phi} = B \tag{15}$$

$$\tilde{\phi} = K^{-1}B \tag{16}$$

Now for solving heat equation inside the human eye, at the first step we must compose the functional for differential heat equation, so we apply boundary conditions and finally we minimize the functional to obtain temperature in each point in human eye. We compose two functional because we have two kinds of boundary conditions in the heat equation for human eye. The functional for corneal surface that has radiation boundary condition is:

$$F(T) = \frac{1}{2}\iiint_v \left[k\left((\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2 + (\frac{\partial T}{\partial z})^2\right)\right] dV + \iint \left(\frac{1}{2}(\alpha_a T\, \varepsilon\sigma T^4) - (\alpha_a T_a + \varepsilon\sigma {T_a}^4 - E)\right) T\, dS_1 \tag{17}$$

The functional for other surfaces that has robin boundary condition is:

$$F(T) = \frac{1}{2}\iiint_v \left[k\left((\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2 + (\frac{\partial T}{\partial z})^2\right)\right] dV + \iint \alpha_{bl}\left(\frac{1}{2}T - T_{bl}\right) T\, dS_2 \tag{18}$$

In the finite element method, the region of interest is descretized, Figure 3 shows the 3D mesh used for modeling the human eye. For minimization the functional in above equation, one needs to get derivation with respect to T. In finite element method for each tetrahedral element in three dimensional model, T is expressed base on interpolation function in following equation.

$$T^e(x, y, z) = \sum_{j=1}^{4} N_j^e(x, y, z) T_j^e \tag{19}$$

In the above equation N_j^e is interpolation function and T_j^e is the temperature of each node in tetrahedral element which must be determined. Now we minimize functional equation. The result of deriving the equation to T for both boundary conditions is:

$$\begin{aligned}\frac{\partial F^e}{\partial T_j^e} &= \sum_{j=1}^{4} T_j^e \iiint_{v^e}\left[k\left(\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y} + \frac{\partial N_i^e}{\partial z}\frac{\partial N_j^e}{\partial z}\right)\right] dV + \sum_{j=1}^{3} T_j^e \iint \alpha_a N_i^e N_j^e dS_1 - \\ &\sum_{j=1}^{3}\iint \left(\alpha_a T_a + \varepsilon\sigma {T_a}^4 - E\right) N_j^e\, dS_1 + \\ &\frac{5}{2}\sum_{j=1}^{3} {T_j^e}^4 \iint \varepsilon\sigma N_i^e {N_j^e}^4\, dS_1 + \\ &\sum_{j=1}^{3} T_j^e \iint \alpha_{bl} N_i^e N_j^e dS_2 + \\ &\sum_{j=1}^{3}\iint (\alpha_{bl} T_{bl}) N_j^e dS_2\end{aligned} \tag{20}$$

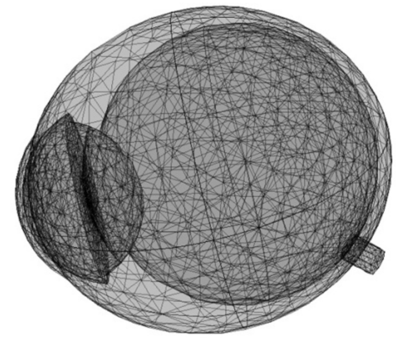

Fig 3. The 3D mesh used for modeling the human eye

In the above equation S_1 refers to each region that has radiation boundary condition and S_2 refers to each region that has robin boundary condition. For each volume we know:

$$\iiint (N_1^e)^k . (N_2^e)^l . (N_3^e)^m . (N_4^e)^n dV = \frac{k!l!m!n!}{(k+l+m+n+3)!} 6V^e \tag{21}$$

The above equation for each surface is:

$$\iint (N_1^e)^k . (N_2^e)^l . (N_3^e)^m . dS = \frac{k!l!m!n!}{(k+l+m+n+2)!} 2S^e \tag{22}$$

In above equations V^e is the volume of each element and S^e is the surface of each element.

Using above equations, for each region that has radiation boundary condition, the equation (20) is simplified to:

$$\frac{\partial F^e}{\partial T_j^e} = [k^e]\{T^e\} + [M^e]\{T^{e4}\} - \{b^e\} \tag{23}$$

The equation (20) for each region that has robin boundary condition is:

$$\frac{\partial F^e}{\partial T_j^e} = [k^e]\{T^e\} - \{b^e\} \tag{24}$$

Assembling the above equations for each element, we can compose final equation in following for our model:

$$\frac{\partial F}{\partial T} = [K]\{T\} + [M]\{T^4\} - \{B\} \tag{25}$$

Minimizing the above equation is equal to:

$$\frac{\partial F}{\partial T} = 0 \tag{26}$$

Therefore we have:

$$[K]\{T\} + [M]\{T^4\} - \{B\} = 0 \tag{27}$$

Without radiation boundary condition, the above equation is in following form:

$$[K]\{T\} - \{B\} = 0 \tag{28}$$

Therefore we have:

$$T = K^{-1}B \tag{29}$$

In this problem we face to nonlinear equation. In last papers the authors model the radiation boundary equation to robin boundary equation and finally they obtain a linear equation like in equation (28), but in this paper we compose the functional for radiation boundary condition and we obtain a nonlinear equation and we must solve this nonlinear equation.

A. Solving nonlinear equation

There are several methods to solve nonlinear equation systems, probably; the most popular techniques are the Newton type techniques [7]. Other techniques are such as Trust-region method, Broyden method. In Newton method, We can approximate the function f using the first order Taylor expansion in a neighborhood of a point $x^k \in \mathcal{R}^n$. The Jacobian matrix $J(x^k) \subset \mathcal{R}^{nxn}$ to f(x) evaluated at x^k is given by:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (30)$$

$$f(x^k + t) = f(x^k) + J(x^k)t + O(\|p\|^2) \quad (31)$$

By setting the right side of the equation to zero and neglecting terms of higher order (except the first) ($O(\|p\|^2)$), we obtain:

$$J(x^k)t = -f(x^k) \quad (32)$$

Then, the Newton algorithm is described as follows:
Set k=0, guess an approximate solution for x^0 and Repeat the following action:
Compute $J(x^k)$ and $f(x^k)$.
Solve the linear system $J(x^k)t = -f(x^k)$
Set $x^{k+1} = x^k + t$ and Set $k = k + 1$.
Until converged to the solution.
The index k is an iteration index and x^k is the vector x after k iterations. The idea of the method is to start with a value which is reasonably close to the true zero, then replace the function by its tangent and computes the zero of this tangent. This zero of the tangent will typically be a better approximation to the function's zero, and the method can be iterated.

B. Implementation

Now after drawing the human eye model, meshing the model, applying boundary conditions and composing the matrices in equation (27), we use Newton method to solve nonlinear equation. As we know, we must define an initial value for all points in our model. The initial temperature in our model is zero for all points, so we use Newton method, and after some iteration we can obtain final value. Now for comparing our results with real results such as results of COMSOL software, we compare the temperature of points on visual axis (light path in eye) of our algorithm with results of COMSOL software. In table II the temperature of 19 points on visual axis for our proposed algorithm and COMSOL software are shown. Now to evaluate our algorithm, the mean error between our algorithm and COMSOL software to COMSOL software values is 0.0075℃, and therefore we can understand the accuracy of our algorithm. Now for better comparison between our results and COMSOL software results, we compare the temperature of each point of our algorithm with COMSOL results, the mean squared error between COMSOL and our algorithm results is 0.0630 and the mean of error between COMSOL software and our algorithm results to COMSOL software results is 0.0052, therefore considering these results, we can understand the accuracy of our algorithm.

Table II. Results of our method and COMSOL software on visual axis

Points	***Our Algorithm***	***FEMLAB***
1	36.9483	36.9537
2	36.9453	36.9478
3	36.9352	36.9378
4	36.9493	36.9404
5	36.9413	36.9389
6	36.9460	36.9365
7	36.8561	36.7803
8	36.6402	36.4593
9	36.3454	35.9860
10	36.4924	36.2804
11	36.6980	36.5560
12	36.3457	35.9865
13	36.3484	35.9855
14	36.3457	35.9865
15	35.3019	34.9861
16	35.4153	34.9636
17	35.6365	34.9443
18	35.5171	34.7788
19	35.3938	34.6101

V. CONCLUSION

In this paper we presented the mathematical solution of radiation boundary condition of heat transfer process in human eye. In last paper, the authors modeled the radiation boundary condition to robin boundary condition which is easier than our method to solve, but it may be have some error. In this paper we show the solution of this boundary condition using finite element method and we solve nonlinear equation which is obtained from finite element method using Newton algorithm. Comparing our results with the results of COMSOL software shows the accuracy of our proposed method.

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