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Denoising and Harmonic Artifacts Rejection for ECG P-Waves by Quadratic Variation Reduction

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***Abstract*— Atrial fibrillation (AF) is a common cardiac arrhythmia related to irregular atrial contractions. Several studies have shown that the analysis of P-waves extracted from ECG signals is helpful in understanding the predisposing factors to AF. However, P-waves are usually highly corrupted by noise and harmonic artifacts and this makes quite difficult their analysis. Recently we proposed a novel algorithm for denoising P-waves based on the notion of quadratic variation reduction. It is quite good in denoising P-waves affected by noise, but its effectiveness reduces when it is used in filtering out harmonic artifacts, like power-line interference. In this paper we propose an algorithm that overcomes this limitation and extends our previous method allowing it to both denoise and reject harmonic artifacts. Simulation results confirm the effectiveness of the approach and highlight its ability to remove both noise and artifacts. The algorithm has reduced computational complexity and this makes it suitable for real-time applications.**

I. INTRODUCTION

The electrocardiogram (ECG) is a measure of cardiac electrical activity generated by depolarization/polarization of the heart muscle during each heart beat. In healthy patients, each beat of the ECG trace consists of a P-wave (atrial depolarization), a QRS complex (left and right ventricles depolarization), a T-wave (ventricles repolarization) and sometimes a U-wave. The correct registration of such waves is fundamental to detect diseases as well as to monitor patients' status. Unfortunately ECG signals are susceptible to electromagnetic interferences and can be contaminated by several kinds of noise such as 50 or 60 Hz power-line interference, electromyographic noise and baseline wander [1]. The power-line interference is a common noise component caused by supply plugs and cables, and sometimes can mask ECG signals, especially the segments having low amplitude like P-waves [1]. Variability in the power-line frequency of about a fraction of Hertz and non-linear transformations operating on signals can make unfeasible the use of simple notch filters for power-line noise removal [2]. Using a band-stop filter does not help in this regard, since it induces distortions in ECG signals and impacts the correct delineation of waves, especially when the power-line frequency is highly variable [3], [4]. Some approaches to power-line noise removal resort to an external reference signal for adaptive cancellation [5].

Atrial Fibrillation (AF) is the most common arrhythmia encountered in the clinical practice and is characterized by irregular and non homogeneous atrial contractions. AF is not a mortal disease but can favor the formation of thrombus at risk to embolize. Several studies show that AF can be detected and predicted analyzing P-waves extracted from ECG traces [6]–[9]. Due to the low signal-to-noise ratio associated with P-waves, this portion of ECG is usually analyzed performing averaging and building a P-wave template [10]. Then, morphological features are extracted from the template. This approach provides a robust measure of the persistent atrial activity but has the unavoidable drawback of losing information on the single beat. However, tracking changes between P-waves is extremely important in improving understanding of the pathophysiological mechanisms of atrial substrates predisposing to AF [9]. Indeed, each P-wave provides important information about the corresponding depolarization pattern throughout the atrial substrate. The analysis of temporal variability of P-waves is possible only if reliable beat-to-beat P-waves are available. This is attainable only when noise and artifacts are effectively removed from each P-wave.

The aim of this investigation is to propose a novel method to denoise heavily corrupted P-waves extracted from ECG recordings, in order to improve their signal-to-noise ratio and allow the robust study of their variability.

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II. RATIONALE

A. The quadratic variation

From the previous section, it is evident that the ability to conduct a meaningful analysis of predisposing factors to AF strongly depends on the availability of reliable P-waves. In this regard, P-waves are reliable if the detrimental effects of noise and artifacts are reduced to an acceptable level.

In the recent work [11], we proposed a novel algorithm that proved being quite good in smoothing P-waves. Its effectiveness reduces when it is used in filtering out harmonic artifacts, like power-line interference. To overcome this limitation, in this paper we propose an algorithm that extends our previous approach and allows to both denoise and reject harmonic artifacts in P-waves. It is based on the following idea. The measured P-wave is affected by noise and artifacts whose effect is to introduce additional "variability" into the observed P-wave with respect to the true one. Thus, provided that we introduce a suitable index of variability, denoising can be performed by searching for a version of the P-wave that is close, in some sense, to the observed one, but has less "variability". In the following we make this idea precise.

The variability of a generic vector can be quantified introducing the following

Definition 1: Given a vector $\boldsymbol{x} = [x_1 \cdots x_n]^{\mathrm{T}} \in \mathbb{R}^n$, the *quadratic variation* of $\boldsymbol{x}$ is defined as

$$[\boldsymbol{x}] \doteq \sum_{k=1}^{n-1} (x_k - x_{k+1})^2 \tag{1}$$

and is denoted by $[\boldsymbol{x}]$.

The quadratic variation is a well-known property used in the analysis of stochastic processes [12]. However, in this context we consider it as a function of deterministic or random vectors.

Introducing the $(n-1) \times n$ matrix

$$\boldsymbol{D} = \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix}, \tag{2}$$

the quadratic variation of $\boldsymbol{x}$ becomes $[\boldsymbol{x}] = \|\boldsymbol{D}\boldsymbol{x}\|^2$, where $\|\cdot\|$ is the Euclidean norm.

The quadratic variation is a consistent index of variability and its use is motivated by the following property. For vectors affected by additive noise, on average it does not decrease and is an increasing function of noise variances. In fact, let $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{w}$, where $\boldsymbol{x}_0$ is a deterministic vector and $\boldsymbol{w} = [w_1 \cdots w_n]^{\mathrm{T}}$ is a zero-mean random vector with covariance matrix $\boldsymbol{K}_w = \mathbb{E}\{\boldsymbol{w}\boldsymbol{w}^{\mathrm{T}}\}$. We do not make any assumption about the distribution of $\boldsymbol{w}$, so the following considerations hold regardless of it. Computing the averaged quadratic variation of $\boldsymbol{x}$ we get

$$\mathbb{E}\left\{\|\boldsymbol{D}\boldsymbol{x}\|^2\right\} = \|\boldsymbol{D}\boldsymbol{x}_0\|^2 + \mathrm{tr}\left(\boldsymbol{D}\boldsymbol{K}_w\boldsymbol{D}^{\mathrm{T}}\right). \tag{3}$$

Note that $\mathrm{tr}\left(\boldsymbol{D}\boldsymbol{K}_w\boldsymbol{D}^{\mathrm{T}}\right) \geq 0$, since it is the trace of a positive semidefinite matrix [13], but in all practical cases the inequality is strict. In fact, we have

$$\mathrm{tr}\left(\boldsymbol{D}\boldsymbol{K}_w\boldsymbol{D}^{\mathrm{T}}\right) = \sum_{k=1}^{n-1} \left(\sigma_k^2 + \sigma_{k+1}^2 - 2\sigma_{k,k+1}\right) \tag{4}$$

where $\sigma_k^2 = \mathbb{E}\{w_k^2\}$ and $\sigma_{k,k+1} = \mathbb{E}\{w_k w_{k+1}\}$. From (4) follows that $\mathrm{tr}\left(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{K}_w\right) = 0$ if and only if all the components of the noise vector $\boldsymbol{w}$ are almost surely equal[1] and that $\mathbb{E}\left\{\|\boldsymbol{D}\boldsymbol{x}\|^2\right\}$ is an increasing function of noise variances.

For example, in typical scenarios $\boldsymbol{w} = \boldsymbol{m} + \boldsymbol{a}$, where $\boldsymbol{m}$ is due to white Gaussian noise whereas $\boldsymbol{a}$ is due to the residual 50 Hz or 60 Hz power-line noise. We may assume $\boldsymbol{m} \sim \mathcal{N}\left(\boldsymbol{0}, \sigma_m^2\boldsymbol{I}\right)$ and $\boldsymbol{a} = [a_1 \cdots a_n]^{\mathrm{T}}$ vector of samples from a harmonic process, i.e., $a_k = A\cos\left[2\pi\frac{f_0}{F_c}(k-1) + \phi\right]$, with A and ϕ independent, ϕ uniformly distributed in $[0, 2\pi)$, $f_0 \in \{50\,\mathrm{Hz}, 60\,\mathrm{Hz}\}$ and F_c being the sampling frequency. Moreover $\boldsymbol{m}$ and $\boldsymbol{a}$ are independent. In this case it is easy to verify that

$$\mathrm{tr}\left(\boldsymbol{D}\boldsymbol{K}_w\boldsymbol{D}^{\mathrm{T}}\right) = \\ = \frac{2\|\boldsymbol{x}_0\|^2(n-1)}{n}\left[\mathrm{SNR}^{-1} + 4\sin^2\left(\pi\frac{f_0}{F_c}\right)\mathrm{SIR}^{-1}\right] \tag{5}$$

where $\mathrm{SNR} = \|\boldsymbol{x}_0\|^2/n\sigma_m^2$ denotes the signal-to-noise ratio and $\mathrm{SIR} = 2\|\boldsymbol{x}_0\|^2/n\mathbb{E}\{A^2\}$ is the signal-to-interference ratio, considering the power-line noise as interference. From (5) it is evident that the average quadratic variation is a decreasing function of SNR and SIR. This supports the rationale behind quadratic variation reduction as a viable criterion for noise reduction. However, in (5) the SIR^{-1} is multiplied by $4\sin^2(\pi f_0/F_c)$ that is less than 1 when $f_0/F_c < 1/6$. For example, when $f_0 = 60\,\mathrm{Hz}$ and $F_c = 2048\,\mathrm{Hz}$ it is about 3×10^{-2}. As a result, low-frequency harmonic artifacts tend to be less attenuated in response to a quadratic variation reduction. Thus, extra conditions must be considered for harmonic artifact rejection.

B. Harmonic artifacts rejection

The approach proposed to effectively reject harmonic artifacts is to exploit quadratic variation reduction *in conjunction* with an additional requirement. The requirement is to make negligible the energy content of harmonic artifacts in the denoised signal.

To quantify it, denote by $\boldsymbol{x} \in \mathbb{R}^n$ a generic $n-$size real vector and let

$$\boldsymbol{X} = \boldsymbol{W}\boldsymbol{x}$$

be its DFT [14], where $\boldsymbol{W} = [w_{h,k}]$ is the DFT matrix, with $w_{h,k} = \frac{1}{\sqrt{n}}\exp\{-j2\pi(h-1)(k-1)/n\}$, $h, k = 1, \ldots, n$. Now, denote by $\widetilde{\boldsymbol{W}}$ the matrix obtained stacking the rows of $\boldsymbol{W}$ corresponding to the harmonic components that we want to reject. Note that since $\boldsymbol{x}$ is a real vector, symmetries occur in its DFT [14] and rows have to be matched in pairs in general.[2] Matrix $\widetilde{\boldsymbol{W}}$ has dimensions $m \times n$, with $m < n$ in general. The quadratic form

$$\left\|\widetilde{\boldsymbol{W}}\boldsymbol{x}\right\|^2 = \boldsymbol{x}^{\mathrm{T}}\widetilde{\boldsymbol{W}}^{\mathrm{H}}\widetilde{\boldsymbol{W}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\mathrm{Re}\left\{\widetilde{\boldsymbol{W}}^{\mathrm{H}}\widetilde{\boldsymbol{W}}\right\}\boldsymbol{x} \tag{6}$$

quantifies the energy content of the harmonic artifacts. In (6) $(\cdot)^{\mathrm{H}}$ denotes the transpose conjugate and $\mathrm{Re}\{\cdot\}$ is the real part.

In the next section we exploit these results and develop an effective algorithm for smoothing P-waves. Smoothing is meant as joint denoising and harmonic artifacts rejection.

III. SMOOTHING P-WAVES

In this section we denote by $\boldsymbol{p}$ the vector collecting samples from the measured P-wave, the one that is affected by noise and artifacts, and by $\boldsymbol{x}$ the corresponding vector after smoothing. Following the line of reasoning presented in the previous section, we determine $\boldsymbol{x}$ solving the following optimization problem

$$\begin{cases} \text{minimize} & \|\boldsymbol{x} - \boldsymbol{p}\|^2 \\ \text{subject to} & \|\boldsymbol{D}\boldsymbol{x}\|^2 \leq a \\ & \|\widetilde{\boldsymbol{W}}\boldsymbol{x}\|^2 \leq b \end{cases} \tag{7}$$

where $\boldsymbol{D}$ is defined in (2), $\widetilde{\boldsymbol{W}}$ in the previous section and a and b are positive constants controlling the degree of smoothness for $\boldsymbol{p}$. Their values are chosen in accordance

[1] That is $w_1 = w_2 = \cdots = w_n$ with probability 1.

[2] Apart from some special cases where single rows are taken.

with the peculiarity of the problem and satisfy $a < \|\boldsymbol{D}\boldsymbol{p}\|^2$ and $b < \|\boldsymbol{p}\|^2$ in order to avoid trivial solutions.[3] Note that we do not need to know in advance the appropriate values for a and b in any particular problem. In fact, as it will be clear later, the solution to the optimization problem (7) can be expressed in terms of two parameters controlling the degree of smoothness, i.e., the quadratic variation of the solution and the energy of artifacts. These parameters are related to the values of a and b in (7). In this way, smoothing can be performed without caring about a and b, by reducing parametrically the quadratic variation of the solution and the energy of artifacts to the desired levels. In general, the optimal values for the controlling parameters can be found as the ones that entail the maximum SNR gain.

Let us consider (7) in more detail. It is a convex optimization problem, since both the objective function and the inequality constraints are convex. As a consequence, any locally optimal point is also globally optimal [15]. Moreover, since the objective function is strictly convex and the problem is feasible the solution exists and is unique. It is possible to prove that the solution to (7) is given by

$$\boldsymbol{x} = \left(\boldsymbol{I} + \lambda \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} + \nu \mathrm{Re}\left\{\widetilde{\boldsymbol{W}}^{\mathrm{H}}\widetilde{\boldsymbol{W}}\right\}\right)^{-1}\boldsymbol{p} \qquad (8)$$

where $\boldsymbol{I}$ is the identity matrix and λ and ν are nonnegative parameters determined by

$$\|\boldsymbol{D}\boldsymbol{x}\|^2 = a \qquad \text{and} \qquad \|\widetilde{\boldsymbol{W}}\boldsymbol{x}\|^2 = b\,. \qquad (9)$$

Note that in (8) the inverse exists for any $\lambda \geq 0$ and $\nu \geq 0$ and when $\lambda = \nu = 0$ no smoothing is performed. It is interesting that the solution to (7) is a linear operator acting on $\boldsymbol{p}$. Moreover, the parameters λ and ν control the degree of smoothing applied to $\boldsymbol{p}$. In particular, even though λ and ν interact, λ mainly controls the quadratic variation of the solution, i.e., the reduction of wideband noise, whereas ν mainly controls the degree of rejection of harmonic artifacts.

A consequence of this is that we do not need to know in advance the value of a and b in (7), as smoothing can be performed according to (8) and λ and ν can be adapted to fulfill some performance criteria. For example, considering the SNR gain[4] as a performance index, λ and ν can be chosen to achieve the maximum gain.

Finally, some remarks on the computational aspects related to the smoothing operation, since matrix inversion is involved in (8). If the size of vector $\boldsymbol{p}$ is large enough, computational problems may arise. Actually this is not an issue for the typical length of vectors representing P-waves, even considering high sampling frequencies. However, when the size of vector $\boldsymbol{p}$ is large, as for whole ECG records, the computational burden, both in terms of time and memory, and the accuracy become serious issues, even for batch processing.

In this regard, the proposed algorithm behaves favorably. It is possible to prove that smoothing using (8) can be performed with complexity $O(n \log n)$ (or $O(n)$ when $\nu = 0$), where n is the size of vector $\boldsymbol{p}$. This property is very important and makes it suitable also for real-time applications. Just to give an idea of how fast the proposed algorithm is, a MATLAB (ver. 7.11) implementation of (8) with $\nu > 0$ running over a PC equipped with 2.5 GHz Core 2 Duo processor, takes about 0.56 s to smooth an ECG record of 10^6 double precision floating point samples.

[3]When $a \geq \|\boldsymbol{D}\boldsymbol{p}\|^2$ and $b \geq \|\boldsymbol{p}\|^2$ the solution is $\boldsymbol{x} = \boldsymbol{p}$ and no smoothing is performed.

[4]This is the ratio between the SNR after and before smoothing.

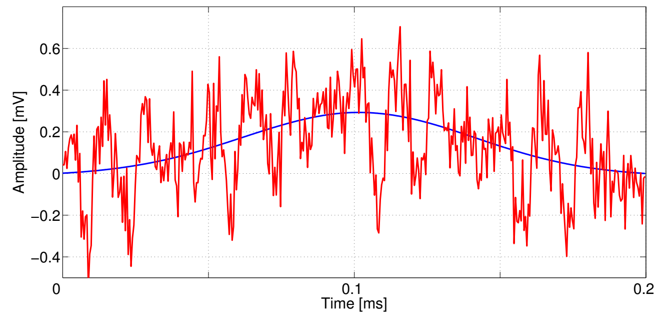

Fig. 1. Noiseless reference P-wave (blue) and noisy P-wave (red).

IV. Simulation results

In this section we test the performance of the proposed algorithm using a reference noiseless model of P-wave. This has been extracted from a synthetic ECG trace generated using the model described in [16]. The P-wave segment has a duration of 200 ms and has been sampled at 2048 Hz. The corresponding samples have been collected in the vector $\boldsymbol{p}_0$. Then, such a reference P-wave $\boldsymbol{p}_0$ has been corrupted by additive noise and harmonic artifacts, denoted by $\boldsymbol{w}$ and $\boldsymbol{d}$ respectively.

The components of $\boldsymbol{w}$ are independent identically distributed zero mean Gaussian random variables with variance σ_w^2 such that $\mathrm{SNR} = 10\log\frac{\|\boldsymbol{p}_0\|^2}{n\cdot\sigma_w^2} = 2\,\mathrm{dB}$, where n is the length of $\boldsymbol{p}_0$. Concerning harmonic artifacts $\boldsymbol{d}$, we considered three sine waves with random phases at frequencies 60 Hz, 80 Hz and 120 Hz respectively. The sine waves at 60 Hz and 120 Hz account for the first and second harmonics of the power-line noise, whereas the 80 Hz sine wave is a generic harmonic interference. Magnitudes of the waves at frequencies 60 Hz and 80 Hz have been chosen to be the same, whereas magnitude of the sinusoid at frequency 120 Hz has been set to half. The resulting signal-to-interference ratio $\mathrm{SIR} = 10\log\frac{\|\boldsymbol{p}_0\|^2}{\|\boldsymbol{d}\|^2}$ was chosen to be 1.5 dB.

Thus, the corresponding noisy P-wave is

$$\boldsymbol{p} = \boldsymbol{p}_0 + \boldsymbol{w} + \boldsymbol{d} \qquad (10)$$

and is characterized by a signal-to-noise-plus-interference ratio $\mathrm{SNIR}_0 = 10\log\frac{\|\boldsymbol{p}_0\|^2}{\|\boldsymbol{w}+\boldsymbol{d}\|^2} = -1.3\,\mathrm{dB}$. Figure 1 shows the noiseless reference P-wave $\boldsymbol{p}_0$ in blue and the corresponding noisy P-wave $\boldsymbol{p}$ of (10) in red.

In Figure 2 we report the reference model ($\boldsymbol{p}_0$, in blue) and the reconstructed wave ($\widetilde{\boldsymbol{x}}$, in red) resulting from a partial smoothing which reduces the quadratic variation of the noisy wave but not the energy of harmonic artifacts, i.e., $\nu = 0$ in

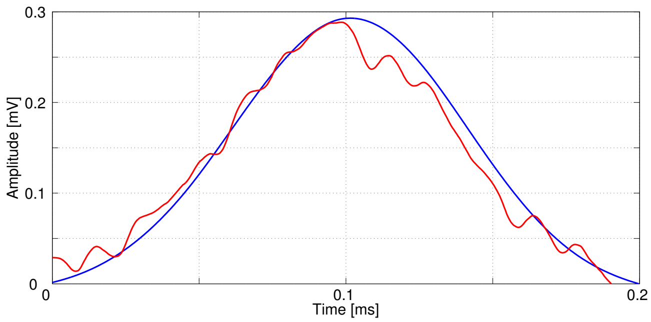

Fig. 2. Reference P-wave $\boldsymbol{p}_0$ (blue) and reconstructed P-wave after partial smoothing $\widetilde{\boldsymbol{x}}$ (red).

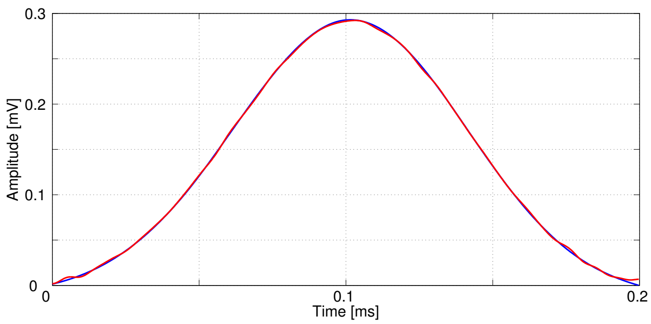

Fig. 3. Reference P-wave $\boldsymbol{p}_0$ (blue) and smoothed P-wave $\boldsymbol{x}$ (red).

(8). The parameter $\lambda = 524$ was chosen in order to minimize the Euclidean distance $\|\widetilde{\boldsymbol{x}} - \boldsymbol{p}_0\|$, where $\widetilde{\boldsymbol{x}} - \boldsymbol{p}_0$ is the error vector with respect to the reference model $\boldsymbol{p}_0$. Although the SNIR in $\boldsymbol{p}$ is quite low, the partial smoothing with $\nu = 0$ and $\lambda = 524$ is very effective in denoising $\boldsymbol{p}$ and the resulting wave $\widetilde{\boldsymbol{x}}$ may be considered a good approximation of $\boldsymbol{p}_0$. Indeed, measuring the performance of the proposed algorithm in terms of SNIR gain

$$\mathrm{G}_{\mathrm{SNIR}} = \mathrm{SNIR}_{\mathrm{S}} - \mathrm{SNIR}_0 = 10 \log \frac{\|\boldsymbol{w} + \boldsymbol{d}\|^2}{\|\widetilde{\boldsymbol{x}} - \boldsymbol{p}_0\|^2} \qquad (11)$$

where $\mathrm{SNIR}_{\mathrm{S}}$ is the SNIR after smoothing and SNIR_0 is the SNIR before smoothing, for these realizations of noise and harmonic interferences we obtain $\mathrm{G}_{\mathrm{SNIR}} = 20.7\,\mathrm{dB}$.

Nevertheless, to reduce the residual oscillatory behavior in $\widetilde{\boldsymbol{x}}$, we considered the combined action of ν and λ. In Figure 3 we compare the reference P-wave $\boldsymbol{p}_0$ (in blue) and the denoised P-wave $\boldsymbol{x}$ (in red) resulting from smoothing with ν and λ such as to minimize the Euclidean distance $\|\boldsymbol{x} - \boldsymbol{p}_0\|$. Here, the smoothing makes it hard to distinguish between the two waves. The resulting gain is $\mathrm{G}_{\mathrm{SNIR}} = 44.3\,\mathrm{dB}$, considering the same realizations of noise and harmonic interferences of Figure 2.

Eventually, it is important to point out that we evaluated the proposed algorithm on different models of P-wave, realizations of noise and disturbance vectors. The resulting gains were all consistent with the ones reported in this work.

V. Conclusions

In this work we considered the problem of noise and harmonic artifacts removal in the P-wave segment of ECG signals. The proposed approach is based on the notion of *quadratic variation* meant as a suitable index of variability for vectors or sampled functions. We developed an efficient smoothing algorithm, which is the closed-form solution to a constrained convex optimization problem. Denoising and harmonic artifacts rejection are achieved by reducing the quadratic variation and the artifacts energy content of the noisy P-waves. The filtered P-wave is obtained as the wave closest to the original one but with reduced quadratic variation and artifact energy content. The computational complexity of the algorithm is $O(n \log n)$ in the size n of the vector to be processed, and this makes it suitable for real-time applications. The algorithm is controlled by only two parameters allowing an easy setup. Simulation results confirm the effectiveness of the approach and highlight its ability in reducing both noise and harmonic artifacts.

The low complexity of the algorithm makes it suitable for processing longer signal records. In this regard we successfully applied it also to unsegmented ECG and EEG traces.

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