Proceedings of the 2012 IEEE 12th International Conference on Bioinformatics
& Bioengineering (BIBE), Larnaca, Cyprus, 11-13 November 2012

Adaptive Nonlinear Probabilistic Filter for Positron Emission Tomography

Musa Alrefaya
Department of Electronics and Informatics
Vrije Universiteit Brussel
Pleinlaan 2, 1050 Elsene
Brussels, Blegium
Email: malrefay@vub.ac.be

Hichem Sahli
Department of Electronics and Informatics
Vrije Universiteit Brussel
Pleinlaan 2, 1050 Elsene
Brussels, Blegium
Email: hsahli@etro.vub.ac.be

***Abstract*—Radiologists face difficulties when reading and interpreting Positron Emission Tomography (PET) images because of the high noise level in the raw-projection data (i.e. the sinogram). The later may lead to erroneous diagnoses. Aiming at finding a suitable denoising technique for PET images, in our first work, we investigated filtering the sinogram with a constraint curvature motion filter where we computed the edge stopping function in terms of edge probability under a marginal prior on the noise free gradient. In this paper, we show that the Chi-square is the appropriate prior for finding the edge probability in the sinogram noise-free gradient. Since the sinogram noise is uncorrelated and follows a Poisson distribution, we then propose an adaptive probabilistic diffusivity function where the edge probability is computed at each pixel. We demonstrate quantitatively and qualitatively through simulations that the performance of the proposed method substantially surpasses that of state-of-art methods, both visually and in terms of statistical measures.**

***Index Terms*—Sinogram Filtering, Adaptive Denoising, PET Filtering, Chi-square Distribution, Poisson Noise.**

I. Introduction

The search for an efficient and applicable image denoising methods is still a challenge especially for images which often taken in poor conditions such as medical images. Filtering techniques based on the use of partial differential equations (PDE) have been extensively studied since the early work of Perona, [1] and Weickert, [6]. They applied the well-known diffusion formula,

$$\frac{\partial u}{\partial t} = u_t = div(g(|\nabla u|)\nabla u) \qquad u(0) = u_0 \quad (1)$$

where u is the smoothed image, u_t is the partial derivative of u with respect to diffusion time t, 'div' denotes the divergence operator and g is the so-called edge stopping or diffusivity function.

In our previous work [12], we have proposed and investigated a Probabilistic Curvature Motion filter for enhancing the PET image by filtering the PET raw data (sinogram) and the results were very promise. The proposed method was initiated based on (i) the curvature motion method [4], and (ii) the probabilistic diffusivity function [3], and considering the following PET images characteristics [12]:

1) The important features in the sinogram are curved structures with high contrast values. These represent the region of interest in the reconstructed PET image, e.g. tumor.
2) The weak edges in the sinogram are the edges that contain low contrast values.
3) The noise in the sinogram is a priori identified as a Poisson noise.

The following denoising scheme has been applied:

$$u_t = g_1(|\nabla u|)u_{vv} + g_2(|\nabla u|)u_{ww} \qquad (2)$$

where the second order Gauge derivatives of the image in the (vv - along the image edges, and (ww -across the image edges) directions are given by:

$$\begin{aligned} u_{vv} &= \frac{u_{xx}u_y^2 - 2u_x u_y u_{xy} + u_{yy}u_x^2}{\left(u_x^2+u_y^2\right)} \\ u_{ww} &= \frac{u_{xx}u_x^2 + 2u_x u_y u_{xy} + u_{yy}u_y^2}{\left(u_x^2+u_y^2\right)} \end{aligned} \qquad (3)$$

In this work, we aim at better enhancing and preserving the significant features of the PET images using nonlinear curvature motion filter with an improved probabilistic edge-stopping function. Considering the singoram characteristics, we propose a probabilistic edge-stopping function based on Chi-square prior for the ideal sinogram gradient with a spatially adaptive algorithm for calculating the prior odd at each pixel. We show that this method is better enhancing and de-noising sinogram data.

The paper is organized as follows, previous related studies are presented in section II. Section III describe the probablisitic diffusivity function briefly. In scetion IV, we present how the Chi-square is the appropriate prior for the ideal sinogram and the adaptive probabilistic filter is illustrated in section V. Finally, section VII conclude the paper.

II. Literature Review

The literature contains many studies investigating a wide range of methods for enhancing PET images. In this section we present some of the related works. Authors in [18] proposed an anisotropic diffusion filter for enhancing and smoothing noisy images where the nonlinear diffusion coefficients were locally adjusted according to the directional derivatives of the image. Authors in [16] proposed a nonlinear PDE-based filtering methods for enhancing and sharpening noisy images by combining backward and forward diffusion processes. A local spatial dependant approach was used for estimating the contrast parameter in the used diffusivity function. In another work [14], authors have proposed a new method where information regarding the local image noise level was used to adjust the amount of denoising strength of the non-local filter. An adaptive nonlinear diffusion (Perona and Malik [1]) filter based on varying the diffusion level according to a local estimation of the image noise was introduced by [15]. The diffusivity function was weighted based on an estimate of the noise level for each pixel. In another study [11], a nonlinear diffusion method for filtering MR images with varying noise levels was presented. Authors assumed that the MR image can be modeled as a piecewise constant (slowly varying) function and it is corrupted by additive zero-mean Gaussian noise.

Pizurica, et al. [7] proposed a probabilistic wavelet shrinkage approach for images denoising. Wavelet domain denoising methods for subband-adaptive and spatially-adapt image denoising have been proposed. The approach of [7] was based on the estimation of the probability that a given coefficient contains a significant noise-free component called "signal of interest". Pizurica, et al. found that the spatially adaptive version of their proposed method yielded better results than the existing spatially adaptive ones. Our proposed diffusivity function is inspired by the above work.

III. The Probabilistic Diffusivity Function:

The main idea of the probabilistic diffusivity function, which was proposed in [3], is to express the diffusivity function as a probability that the observed gradient presents no edge of interest under a suitable marginal prior distribution for the noise-free gradient histogram.

Lets first define the formula of the probabilistic edge stopping function. Let m denote the ideal, noise-free gradient magnitude and define the following two hypotheses: H_0 : "an edge element of interest is absent" and H_1 :"edge element of interest is present" precisely as: $H_0 : m \leq \sigma,$ and $H_1 : m > \sigma.$
The noise level σ is estimated using wavelet based method where the noise is reconstructed from wavelet coefficients at the finest level of detail, as presented in [12]. The diffusivity function is defined as:

$$g(x) = A(1 - P(H_1|x)) \tag{4}$$

Where A is a normalizing constant, and it is chosen as:$A = 1/(1-P(H1|0))$. The Laplacian or double exponential prior was considered: $p(Y) = \frac{\lambda}{2}e^{-\lambda|y|}$. In [3] it has been demonstrated that the Bayes' rule yields $P(H1|X) = \mu\eta(X)/[1+\mu\eta(X)]$ where $\mu = P(H_1)/P(H_0)$ is the *prior odds*, and $\eta(X) = p(X|H1)/p(X|H0)$ is the likelihood ratio. The diffusivity function 4 becomes:

$$g_{pr}(X) = (1 + \mu\eta(0))\frac{1}{1 + \mu\eta(X)} \tag{5}$$

IV. A Chi-square Prior for Ideal Sinogram Gradient

Most of the proposed image denoising algorithms assume that the noise is normally distributed and additive. Many images, such as those from radiography, contain noise that satisfies a Poisson distribution. The magnitude of Poisson noise varies across the image, as it depends on the image intensity. This makes removing such noise very difficult. In this section, we show that the Chi-square is the appropriate prior for the ideal sinogram gradient. First, we demonstrate that a Poisson distribution can be approximated by a Gaussian distribution. The probability mass function of the Poisson distribution is given as: $P(S) = \frac{m^S e^{-m}}{S!}$

Where S is the number of occurrences of an event and m is the expected number of occurrences during a given interval.

In the literature [8–10], several studies presented and discussed that the Poisson distribution approaches a Gaussian density function in the case of high number of counts. Moreover, Miller et al. [8] showed that the Gaussian approximation is surprisingly accurate, even for a fairly small number of counts. For showing that a Poisson distribution can be approximated by a Gaussian distribution, we use the logarithmic function to simplify the proof:

$$\ln P(S) = \ln\left(\frac{m^S e^{-m}}{S!}\right) \tag{6}$$

Using Stirlings formula (for large S as we assuming here)

$S! \approx S^S.e^{-S}\sqrt{2\pi S}$

we have

$$P(S) \approx \frac{e^{-(S-m)^2/2m}}{\sqrt{2\pi m}} \tag{7}$$

In the following, we demonstrate that the gradient of Poisson random variables follows a Skellam distribution and the Skellam distribution can be approximated as Gaussian distribution.

Assuming that the sinogram gradient is approximated by absolute difference of neighboring pixel values on a 2-connected grid. The distribution of the difference of two statistically independent random variables $s1$ and $s2$, $a = s1 - s2$, each having Poisson distributions with different expected values m_1 and m_2, is denoted as the Skellam distribution [17], and can be given as:

$$PD(a; m_1, m_2) = e^{-(m_1+m_2)}(\frac{m_1}{m_2})^{a/2} I_{|a|}(2\sqrt{\mu_1\mu_2}) \tag{8}$$

where $I_k(Z)$ is the modified Bessel function of the first kind.

Lets $s1$ and $s2$ are two statistically independent adjacent pixels in the observed sinogram follows a Gaussian distribution with means m_1 and m_2, as follows: $s1 \sim Gauss(m_1, \sigma_{s1})$ and $s2 \sim Gauss(m_2, \sigma_{s2})$. The difference between two Poisson variables has the following properties: $i)\sigma^2_{s1s2} = \sigma^2_{s1} + \sigma^2_{s2} = 2\sigma^2$ and $2)m = m_{s1s2} = m_{s1} - m_{s2} = 0$. Considering that properties, the cross-correlation and the delta function, the approximated distribution of the sinogram gradient can be given as $Gauss(0, \sqrt{2m})$.

Based on the assumption that the sinogram gradient follows the $Gauss(0, \sqrt{2m})$ distribution, we can show that the distribution of this gradient leads a Chi-square distribution as follows,

$$\begin{aligned} \nabla S_{(i,j)} &\sim Gauss(0, \sqrt{2m}) \\ \frac{|\nabla S_{(i,j)}|}{\sqrt{2m}} &\sim Gauss(0,1) \\ \frac{|\nabla S_{(i,j)}|^2}{2m} &\sim \chi^2 \end{aligned} \tag{9}$$

The Chi-square distribution is defined by the following probability density function:

$$P(y) = \frac{y^{\zeta/2-1}.e^{-y/2}}{2^{\zeta/2}\gamma(\zeta/2)} \tag{10}$$

where $\gamma(\zeta/2)$ denotes the Gamma function, and ζ is a positive integer that specifies the number of degrees of freedom. For the noise gradient model $x = y + n$, the chi-square with 2 degrees of freedom (2 degrees because we are dealing with 2D images), is given as:

$$P(y) = \frac{1}{2}.e^{-\frac{1}{2}|y|} \tag{11}$$

Based on the above, the formulas of the Bayesian edge stopping function 4 can be reformulated considering chi-square prior instead of Laplacian prior and $n \sim Gauss(0, 2\sigma_n^2)$.

V. Spatially adaptive Bayesian Diffusivity Function

The probabilistic curvature motion filter with a diffusivity function that consider global parameter (constant threshold), presented in [12] does not consider the images with spatially varying noise levels such as sinograms. The diffusivity function used in [12] has a global threshold parameter which is related to the image noise standard-deviation $T = \sigma_n$. In such formulation, if two pixels/voxels have equal gradient magnitude, they will give the same $g_{pr}(x)$ values, no matter the noise level at these pixels.

In this work, the probabilistic diffusivity function is improved by considering the local statistical noise at each element. We adopt the estimator to the local spatial context in the image following the approach of Pizurica, et al. [7], which was applied in the wavelet domain where here it is used in the spatial domain. The most appropriate way to achieve such a spatial adaptation is to estimate the prior probability of signal presence $p(H_1)$ adaptively for each element instead of fixing it globally. This can be achieved by conditioning the hypothesis H_1 on a local spatial activity indicator such as the locally averaged magnitude or the local variance of the observed gradient.

To estimate the probability that "signal of interest" is present at the position i, we consider a local spatial activity indicator at each position, which is denoted as z_i.

Starting from the prior odd formula $\mu = P(H_1)/P(H_0))$, we replace the ratio of "global" probabilities $p(H_1)/p(H_0)$ with locally adaptive prior, $p(H_1|z_i)/p(H_0|z_i)$, i.e. $p(H_1)$ and $p(H_0)$ are conditioned on the local spatial indicator, xi is the local liklihood ratio,

$$\frac{P(H_1|z_i)}{P(H_0|z_i)} = \frac{P(z_i|H_1)}{P(z_i|H_0)}.\frac{P(H_1)}{P(H_0)} = \xi_i(z_i).\mu \quad (12)$$

where

$$\xi_i(z_i) = \frac{P(z_i|H_1)}{p(z_i|H_0)} \quad \mu = \frac{P(H_1)}{P(H_0)} \quad (13)$$

The local spatial activity indicator z_i is deffined as the locally averaged magnitude of the observed gradient-magnitude elements in a relatively small square window. Considering the Bayes' rule, the probability that "edge of interest" is present at position i, $(P(H_1)|x_i)$, is given as,

$$P(H_1|x_i) = \frac{\hat{\mu}\eta_i(x_i)}{1 + \hat{\mu}\eta_i(x_i)} = \frac{\mu\eta_i(x_i)\xi_i(z_i)}{1 + \mu\eta_i(x_i)\xi_i(z_i)} \quad (14)$$

The spatially adaptive diffusivity function then can be formulated as:

$$\begin{aligned} g(x_i, z_i) &= A(1 - P(H_1|x_i, z_i)) \\ g(x_i, z_i) &= (1 + \mu\eta(0))(\tfrac{1}{1+\mu\eta_i(x_i)\xi_i(z_i)}) \end{aligned} \quad (15)$$

where

$$\eta_i(x_i) = \frac{P(x_i|H_1)}{P(x_i|H_0)} \quad (16)$$

We ensure that $g(0) = 1$, because the minimum of $P(H1|x)$ is at $x = 0$ and thus $(1 - P(H_1|x))$ peaks at $x = 0$.

Intuitively, the proposed method consider an 'observed gradient' at a given location as how how probable is this location presents useful information compared to its neighborhood, based on:
1- The likelihood ratio via $\eta_i(x_i)$
2- A measurement from the local surrounding via $\xi_i(z_i)$
3- Based on the global statistical properties of the elements (via μ).

The local spatial activity indicator (z_i) is defined as the locally averaged magnitude of the observed gradient-magnitude elements in a relatively small square window $w(i)$ of a fixed size N:

$$z_i = \frac{1}{N} \sum_{l \in w(i)} M_l \quad (17)$$

where M_l is the gradient magnitude at positions $l \in w(i)$.

Assuming that all the elements within the small window are equally distributed and conditionally independent. With these simplifications, the conditional probability of z_i given H_1 in a square window $w(i)$ of size N, which is denoted as $P_N(z_i|H_1)$, is given by N convolutions of $P(M_i|H_1)$ with itself, as follows

$$P_N(z_i|H_1) = ((P(M_i|H_1)Conv_N(P(M_i|H_1)) \quad (18)$$

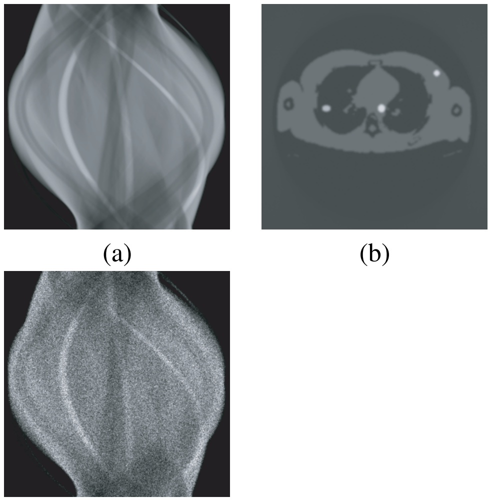
while the conditional probability $P_N(z_i|H_0)$ of z_i given H_0 in $w(i)$ of size N, is given by N convolutions of $P(M_i|H_0)$ with itself,

$$P_N(z_i|H_0) = ((P(M_i|H_0)Conv_N(P(M_i|H_0)) \quad (19)$$

VI. Experimental Results

The validation of the proposed filter employs simulated PET data of a slice of the thorax, which allows generating multiple realizations of the noisy data. Subsequently one can calculate bias and variance and attain a better quantifiable analysis. The simulated PET contains three regions of interest (tumors). We generate 100 realizations with 1x10^6 coincident events. The realization sinogram size is 256x256 pixels and their spacing is 2x2$\frac{\text{mm}}{\text{pixel}}$. The filtered back projection (FBP) is used for reconstructing the PET images. In our experiments we also included datasets where additional noise had been added to the original noise-free image. Figure 1(a)-(b) shows the ideal noise-free sinogram and the reconstructed PET image. A contaminated sinogram is shown in Figure 1(c).

Two types of quantitative evaluation measures are adopted. The first set stems from measuring the quality of the filtering techniques whilst the second set originates from validating the quality of the PET reconstruction. Note that both measurement set require the present of ground-truth data. The former uses the noise-free image, whilst the latter needs prior identification of the important areas by a medical professional. In this work, we present the results of the first method while the results of the second one will be presented in a coming paper.



(a) (b)

(c)

Figure 1. (a) The original -noise free- simulated sinogram and (b) its reconstructed PET image with 3 clearly visible spots (tumors) using (Filtered Back Projection) method.(c) An example of one realization of a noisy sinogram

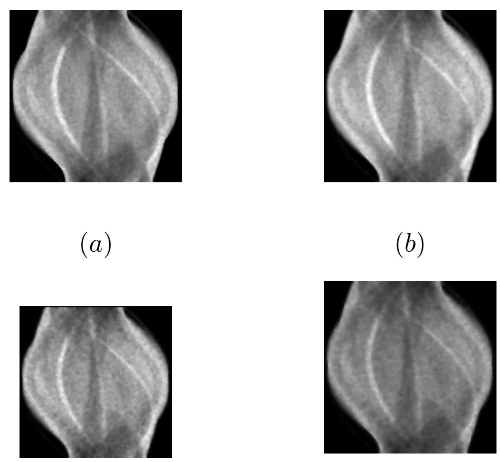
1) Denoising Quality: The idea is to verify the quality of the denoised sinogram. The latter can be easily achieved because of the presence of its noise-free counterpart. In [5], several approaches for measuring the closeness to the noise-free image were analyzed. In this work, we adopt the following measures for evaluating the quality of the diffused sinogram: DQ1

1) The Peak Signal to Noise Ratio Obviously, one sees that the higher the PSNR, the better the quality. $PSNR(t) = 10\log_{10} \frac{\mathrm{Card}(\Omega)}{\sum_{p \in \Omega} |I(p)-u(p,t)|}$
2) The correlation ($\mathcal{C}_{\mathbf{m}\rho}$) between the noise-free and the filtered image. The higher this correlation the better the quality is. $\mathcal{C}_{m\rho}(t) = \rho\left[I, u(t)\right]$
3) The calculated variance of the noise (**NV**) describes the remaining noise-level. Therefore, it should be as small as possible. $\mathrm{NV}(t) = \sigma^2\left[|I - u(t)|\right]$

In this work, we are interested in comparing different filtering approaches: The probabilistic Curvature Motion denoted as PCM [12], the Noise-Adaptive Nonlinear Diffusion Filtering (NAF)[11], the Constraint Curvature Motion Filter, denoted as (CCM)[2], and the proposed method - Adaptive Modified Probabilistic Curvature Motion, denoted as MAPCM. The later yielded the best obtainable result for all the 3 considered error measures applied to an optimal selected selected scale s_opt obtained using the maximum correlation method of [5].

However, since these measure commonly do not agree upon a single best scale, we are more concerned with the comparison at the selected scale t_{opt}.we use an earlier proposed optimal scale selection approach [5], where the maximum correlation method has been adopted. The free parameters in the CCM and NAF filter are empirically tuned to give the best results.

Figure 2 shows examples of a filtered realizations by the proposed filter (MAPCM), PCM, NAF and CCM-sapiro filters, respectively. Table I shows the results of filtering 100 noisy sinograms using the normal probabilistic curvature motion (PCM) filter, the proposed filter (MAPCM) filter, CCM-sapiro and NAF filters from the literature.



(a) (b)

(c) (d)

Figure 2. Filtered sinogram:(a) an example of a filtered sinogram by PCM filter, (b) MAPCM filter, (c)CCM and (d)NAF filters

$Method$	F	PCM	$MAPCM$	CCM	NAF
$PSNR$	15.08	29.73	**30.84**	27.4	29.1
NV	0.1087	0.0223	**0.0200**	0.0244	0.0261
$\mathcal{C}_{m\rho}$	0.920	0.9950	**0.9955**	0.9942	0.9911

Table I

DENOISING QUALITY MEASURES FOR THE NOISY SINOGRAM (F), THE FILTERED REALIZATIONS BY PCM, MAPCM, CCM AND NAF FILTERS THE BEST PERFORMING FILTERING METHOD PER MEASURE IS DISPLAYED IN BOLD.

Results of the MAPCM filter demonstrate that the Chi-square distribution is an appropriate marginal prior for the sinogram noise-free gradient for computing edge element of interest. Additionally, The AMPCM is more effective for preserving the edges and enhancing sinograms as it considered the prior odd at each position. This filter has proven to give better PSNR results than filters from the literature PCM, CCM and NAF as shown in TableI. It preserves the boundaries of the curvy shape features and wisely smoothes the regions of interest as well as the other regions as shown in Figure 2. The high level of Poisson noise in the sinogram affect the performances of the CCM and NAF filters since the diffusivity function are not estimated well.

The drawback of CCM method is the introduction of a free parameter k, which penalizes the edge strength since it has the same value for all pixels. On the other hand, the main drawback of the NAF filter, with respect to the Poisson noise, which characterize sinogram images, is that the diffusion produces important oscillations in the gradient, which finally leads to a poorly smoothed image, as shown in figure 2-d.

VII. Conclusion

Adaptive probabilistic curvature motion filter for enhancing PET images is developed and discussed in this work. The filter is applied on the 2D sinogram pre-reconstruction. For considering the special characteristics of the sinogram data and the Poisson noise, a Chi-square is used as a marginal prior for ideal sinogram gradient in the diffusivity function. We showed that this prior is more appropriate for the sinogram noise-free gradient. Using the adaptive probabilistic diffusion function has proven to be an effective and suitable tool for controlling the diffusion process in the proposed scheme. It demonstrates better results and show that this function is more suitable for the sinogram data with regards to smoothing the noise level.

Acknowledgment

The authors sincerely wishes to express great thanks to Prof. M. Defrise, Division of Nuclear Medicine at AZ-VUB, for the fruitful discussions and feedback.

References

[1] P. Perona and J. Malik, "Scale space and edge detection using anisotropic diffusion", IEEE Trans. on Pattern Analysis and Machine Intelligence, vol. 12, pp.629-639, 1990.

[2] L. Alvarez, P. Lions and J. Morel, "Image selective smoothing and edge detection by nonlinear diffusion", SIAM Journal on Numerical Analysis Vol. 29, pp. 845866, 1992.

[3] A. Pizurica, I. Vanhamel, H. Sahli, W. Philips, and A. Katartzis, "A Bayesian formulation of edge-stopping functions in non-linear diffusion", IEEE Signal Processing Letters, vol. 13(8), pp. 501-504, 2006.

[4] G. Sapiro, Geometric partial differential equations and image analysis, University Press, Cambridge, 2001.

[5] I. Vanhamel, C. Mihai, H. Sahli, A. Katartzis, I. Pratikakis, "Scale Selection for Compact Scale-Space Representation of Vector-Valued Images", Int. J. of Computer Vision, vol. 84(2),pp.194-204, 2008.

[6] J. Weickert, Anisotropic diffusion in image processing, ECMI Series. Teubner-Verlag, Stuttgart, Germany, 1998.

[7] A. Pizurica, P. Scheunders, and W. Philips, "Multiresolution multispectral image denoising based on probability of presence of features of interest", Advanced Concepts in Image and Vision Systems, PP.357-364, 2004.

[8] G. Miller, H. Martz, TT. Little, and R. Guilmette, "Using exact Poisson likelihood functions in Bayesian interpretation of counting measurements", Health Phys, vol.83(4), pp.512-518, 2002.

[9] A. Brodsky, "Exact calculation of probabilities of false positives and false negatives for low background counting", Health Phys, vol. 63(2),pp.198-204,1992.

[10] A. Gersch, "Simple formula for the distortions in a Gaussian representation of a Poisson distribution", American J. of Phys., Vol 44(9), pp. 885-886, 1976.

[11] A. Samsonov and C. Johnson, "Noise-adaptive nonlinear diffusion filtering of MR images with spatially varying noise levels", Magnetic Resonance in Medicine vol.52, pp.798806, 2004.

[12] M. Alrefaya, H. Sahli, and I. Vanhamel, D. Hao, "A Nonlinear Probabilistic Curvature Motion Filter for Positron Emission Tomography Images", Scale Space and Variational Methods in Computer VisionLNCS, from Lecture Notes in Computer Sciencevol.5567, pp.212 - 223, 2009.

[13] I. Vanhamel, "Vector valued nonlinear diffusion and its application to image segmentation", Ph.D. dissertation, Vrije Universiteit Brussel, 2006.

[14] J. Manjon, P. Coupe, L. Mart-Bonmat, D. Louis Collins, and M. Robles, "Adaptive Non-Local Means Denoising of MR Images With Spatially Varying Noise Levels", Jorunal of Magnatic Resonance Imaging, vol.31, pp.192203, 2010.

[15] R. Padfield, and R. Manjeshwar, "Adaptive conductance filtering for spatially varying noise in PET images", Progress in biomedical Optics and imaging, vol.7(3), 2006.

[16] G. Gilboa, N. Sochen, and Y. Yehoshua, "Forward-and-Backward Diffusion Processes for Adaptive Image Enhancement and Denoising", IEEE Trans. on Image Processing, vol.11(7), 2002.

[17] J. Skellam, "The frequency distribution of the difference between two Poisson variates belonging to different populations", Journal of the Royal Statistical Society: Series A, vol. 109 (3), pp.296, 1946.

[18] S. Fu, Q. Ruan and W. Wang, "Adaptive Bidirectional Flow for Image Interpolation and Enhancement", Int. Journal of IT, vol.2(4), 2004.

[19] Y. Wang, R. Niu, L. Zhang, K. Wu and H. Sahli, "A scale-based forward-and-backward diffusion process for adaptive image enhancement and denoising", EURASIP Journal on Advances in Signal Processing, Vol. 22, 2011.