Assessing instantaneous energy in the EEG: a non-negative, frequency-weighted energy operator

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Abstract— **Signal processing measures of instantaneous energy typically include only amplitude information. But measures that include both amplitude and frequency do better at assessing the energy required by the system to generate the signal, making them more sensitive measures to include in electroencephalogram (EEG) analysis. The Teager–Kaiser operator is a frequency-weighted measure that is frequently used in EEG analysis, although the operator is poorly defined in terms of common signal processing concepts. We propose an alternative frequency-weighted energy measure that uses the envelope of the derivative of the signal. This simple envelope–derivative operator has the advantage of being non-negative, which when applied to a detection application in newborn EEG improves performance over the Teager–Kaiser operator: without post-processing filters, area-under the receiver-operating characteristic curve (AUC) is 0.57 for the Teager–Kaiser operator and 0.80 for the envelope–derivative operator. The envelope–derivative operator also satisfies important properties, similar to the Teager–Kaiser operator, such as tracking instantaneous amplitude and frequency.**

I. INTRODUCTION

Energy is a difficult term to define in a signal processing context. The signal processing definition differs from the definition used in physics, which is a measure of work done (or work that can be done) in a system, because we often don't know or don't have access to the system generating the signal. For example, the signal processing definition assesses amplitude only and assigns the same value to two unit-amplitude signals, one at 1 Hz and the other at 1 000 Hz, even though the energy (work done) to generate these signals can differ.

Addressing this inadequacy, Kaiser proposed an energy measure, based on previous unpublished work by Teager, that includes not only the amplitude but also the frequency of the signal [1]. Using this Teager–Kaiser definition, the unit-amplitude signals at different frequencies show different energy. This definition, often referred to as the nonlinear-energy operator, also differs from the classical energy measure because it is an instantaneous measure; that is, it is a function of time and can track changes in signal—and therefore system—energy.

Computer code to implement the methods presented here is available at http://otoolej.github.io/code/nleo/

This Teager–Kaiser measure has been applied in many areas of biomedical signal processing, including electroencephalogram (EEG) analysis [2], [3]. A limitation of this Teager–Kaiser operator is interpretation: the measure is the output of a nonlinear system that includes a second-order differential equation. And in most EEG applications, there is significant post-processing of the operator which casts some doubt on the applicability of this measure. We propose to study the Teager–Kaiser operator from a signal processing prospective and test our conclusions on a data set of EEG recorded from newborn infants.

II. FREQUENCY-WEIGHTED ENERGY

Kaiser proposed a measure to assess the instantaneous energy of the signal that incorporates both the amplitude and frequency of the signal [1]. The measure, for continuous-time signal $x(t)$, is defined as a second-order differential equation [4]

$$\Psi[x(t)] = \dot{x}^2(t) - x(t)\ddot{x}(t) \qquad (1)$$

using the notation $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$ and $\ddot{x}(t) = \mathrm{d}^2x(t)/\mathrm{d}t^2$. For signal $x(t) = A\cos(\omega_0 t + \phi)$, $\Psi[x(t)] = A^2\omega_0^2$, a frequency-weighted energy measure which Kaiser relates to the physical energy (work done) in generating simple harmonic motion in a mechanical system [1], [4].

Typical signal processing measures of instantaneous energy are the amplitude square of the signal, that is $|x(t)|^2$, or the envelope of the signal,

$$S[x(t)] = |x(t) + \mathrm{j}\mathcal{H}[x(t)]|^2 \qquad (2)$$

where $\mathcal{H}[.]$ is the Hilbert transform. The envelope quantifies energy in terms of amplitude, for example $S[A\cos(\omega_0 t + \phi)] = A^2$, and is independent of frequency.

A. Proposed Energy Measure

Frequency information can be included in the envelope measure from (2) by first applying a weighting filter, with frequency response $|H(\omega)|^2 = \omega^2$, to the signal. Maintaining similarity with the Teager–Kaiser operator, we select the derivative function as the filter, using the property that the Fourier transform of $\dot{x}(t)$ is $\mathrm{j}\omega X(\omega)$, where $X(\omega)$ is the Fourier transform of $x(t)$. We thus define the operator as

$$\Gamma[x(t)] = |\dot{x}(t) + \mathrm{j}\mathcal{H}[\dot{x}(t)]|^2 = \dot{x}^2(t) + \mathcal{H}[\dot{x}(t)]^2 \qquad (3)$$

And importantly,

$$\Gamma[A\cos(\omega_0 t + \phi)] = A^2\omega_0^2$$

This proposed measure is simply the combination of filtering, to weight higher-frequency components, with the time-varying envelope of the signal. We refer to this measure as the *envelope–derivative operator.*

Although the Teager–Kaiser and the proposed operators are very similar, and the first terms of (1) and (3) are equal, they do differ in their seconds terms. The difference is best highlighted in the frequency domain, as

$$-\mathcal{F}\left\{x(t)\ddot{x}(t)\right\} = \omega^2 X(\omega) * X(\omega)$$
$$\mathcal{F}\left\{\mathcal{H}\left[\dot{x}(t)\right]^2\right\} = |\omega|X(\omega) * |\omega|X(\omega)$$

where $*$ represents the convolution operation and using the identity $\mathcal{F}\left\{\mathcal{H}[x(t)]\right\} = -\mathrm{j}\,\mathrm{sgn}(\omega)X(\omega)$.

B. Properties

Although the two operators differ they have similar properties. The following presents a brief outline of proofs of the properties presented by Kaiser [1] for the proposed envelope–derivative operator. First, as mentioned previously, for $x(t) = A\cos(\omega_0 t + \phi)$,

$$\Gamma[x(t)] = A^2\omega_0^2\left[\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)\right]$$
$$= A^2\omega_0^2$$

as $\dot{x}(t) = -A\omega_0\sin(\omega_0 t + \phi)$ and $\mathcal{H}[\dot{x}(t)] = A\omega\cos(\omega_0 t + \phi)$.

For amplitude-modulated signal $x(t) = A\mathrm{e}^{rt}\cos(\omega_0 t + \phi)$, where there is no spectral overlap between e^{rt} and $\cos(\omega_0 t + \phi)$, then

$$\Gamma[x(t)] = A^2\mathrm{e}^{2rt}r^2\left[\cos^2(\omega_0 t + \phi) + \sin^2(\omega_0 t + \phi)\right]$$
$$+ A^2\mathrm{e}^{2rt}\omega^2\left[\cos^2(\omega_0 t + \phi) + \sin^2(\omega_0 t + \phi)\right]$$
$$= A^2\mathrm{e}^{2rt}(\omega^2 + r^2)$$

as $\dot{x}(t) = A\mathrm{e}^{rt}[r\cos(\omega_0 t + \phi) - \omega\sin(\omega_0 t + \phi)]$ and $\mathcal{H}[\dot{x}(t)] = A\mathrm{e}^{rt}[r\sin(\omega_0 t + \phi) + \omega\cos(\omega_0 t + \phi)]$. The operator therefore tracks the time-varying amplitude e^{2rt}.

For a frequency-modulated signal $x(t) = A\cos[\phi(t)]$ with $\phi(t) = \omega_0 + \int_o^t f(\tau)\mathrm{d}\tau$, we have

$$\Gamma[x(t)] = A^2[\omega_0 + f(t)]^2\{\cos^2[\phi(t)] + \sin^2[\phi(t)]\}$$
$$= A^2[\omega_0 + f(t)]^2$$

as $\dot{x}(t) = -A[\omega_0 + f(t)]\sin[\phi(t)]$ and $\mathcal{H}[\dot{x}(t)] = A[\omega_0 + f(t)]\cos[\phi(t)]$, assuming that there is no spectral overlap between the instantaneous frequency (IF) $\omega_0 + f(t)$ and $\cos[\phi(t)]$. The operator therefore tracks the IF of the signal.

For a linear combination of two signals, the operator (like the Teager–Kaiser operator) contains cross-terms. Consider $y(t) = x_1(t) + x_2(t)$, with $x_1(t) = A_1\cos(\omega_1 t + \phi_1)$ and $x_2(t) = A_2\cos(\omega_2 t + \phi_2)$, then

$$\Gamma[y(t)] = \Gamma[x_1(t)] + \Gamma[x_2(t)] + a\,[\sin(\omega_1 t + \phi_1)$$
$$\times \sin(\omega_2 t + \phi_2) + \cos(\omega_1 t + \phi_1)\cos(\omega_2 t + \phi_2)]$$
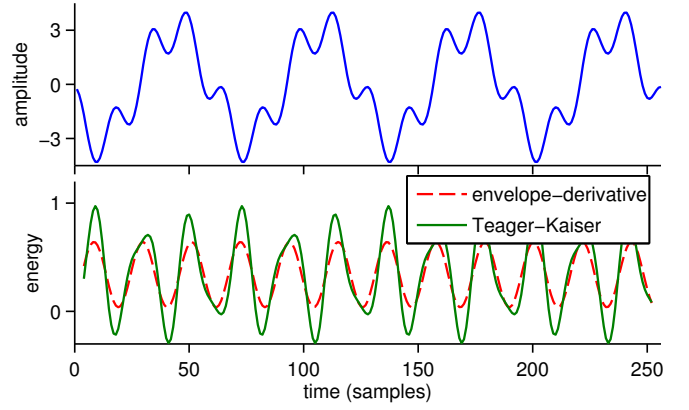$$= \Gamma[x_1(t)] + \Gamma[x_2(t)] + a\cos[(\omega_1 - \omega_2)t + \phi_1 - \phi_2]$$

Fig. 1. Signal with two components. Top plot: test signal $x(n) = 1.3\cos(n\pi/8) + 3.1\cos(n\pi/32)$. Bottom plot: Teager–Kaiser operator and the proposed envelope–derivative operator energy values equal to sum of the individual frequency-weighted energy plus an oscillation term proportional to the difference in the frequencies of $3\pi/32$. The Teager–Kaiser has an additional modulation term which can create negative values.

where $a = 2A_1A_2\omega_1\omega_2$. The operator includes a term which oscillates at a frequency equal to the difference of the two components. This differs to the Teager–Kaiser operator, which includes an additional modulation term [1]. Fig. 1 shows an example comparing the two operators for a two-component signal and highlights the non-negative property of the envelope–derivative operator.

C. Discrete Definition

The discrete version of the continuous equation in (1) is defined as [1]

$$\Psi[x(n)] = x^2(n) - x(n+1)x(n-1) \qquad (4)$$

for discrete signal $x(nT)$, assuming that the sampling period $T = 1$, using the forward difference method to estimate differentiation, with $\dot{x}(n) = x(n+1) - x(n)$. Properties for the discrete version are similar to the continuous definition with one notable distinction: because the forward difference method only approximates the continuous derivative function, $\Psi[A\cos(\omega_0 n)] = A^2\sin^2(\omega_0)$ which differs from the ideal response of $A^2\omega_0^2$. If the sampling frequency is increased by a factor of 4, then $\Psi[A\cos(\omega_0 n)] \approx A^2\omega_0^2$ as $\omega^2 \approx \sin^2(\omega)$ within the region $0 \leq \omega \leq \pi/4$; this approximation holds with a maximum error of 11% [1].

Similarly, using the central-finite difference method, defined as $\dot{x}(n) = [x(n+1) - x(n-1)]/2$, for the envelope–derivative operator in (3),

$$\Gamma[x(n)] = \frac{1}{4}\left[x^2(n+1) + x^2(n-1) + h^2(n+1) + \right.$$
$$h^2(n-1)] + \frac{1}{2}\left[x(n+1)x(n-1) + h(n+1)h(n-1)\right]$$

with $h(n)$ defined as $h(n) = \mathcal{H}[x(n)]$. This discrete Hilbert transform is defined for signal $x(n)$ as $\mathrm{IDFT}\{-\mathrm{j}\,\mathrm{sgn}(N/2 - k)\,\mathrm{sgn}(k)X(k)\}$, where $X(k) = \mathrm{DFT}\{x(n)\}$ (DFT is the discrete Fourier transform and

IDFT is the inverse discrete Fourier transform). Also, similar to the Teager–Kaiser operator, $\Gamma[A\cos(\omega_0 n)] = A^2\sin^2(\omega_0)$.

D. Noise Analysis

The Teager–Kaiser operator is biased in the presence of noise [1]. For signal $y(n) = s(n) + w(n)$, where $w(n)$ is zero-mean white Gaussian noise of variance σ^2 and $s(n)$ is a deterministic signal, taking the expectation gives

$$\mathbb{E}\{\Psi[y(n)]\} = \Psi[y(n)] + \sigma^2$$

where the symbol $\mathbb{E}$ represents the expectation operation. Agarwal and Gotman presented an alternative operator to the overcome this noise-bias [6]:

$$\Theta[x(n)] = x(n-1)x(n-2) - x(n)x(n-3) \quad (5)$$

which does not contain the bias term σ^2,

$$\mathbb{E}\{\Theta[s(n) + w(n)]\} = \Theta[s(n)]$$

The operator in (5) was proposed as part of a more general (discrete) nonlinear operator proposed by Plotkin and Swamy [7] as

$$O[x(n)] = x(n-l)x(n-p) - x(n-q)x(n-s)$$

for $l + p = q + s$. When $l = p$ and $|q - s| = 2$, $O[x(n)]$ is a time-shifted version of the Teager–Kaiser operator. Likewise, when $|l - p| = 1$ and $|q - s| = 3$, $O[x(n)]$ is time-shifted version of $\Theta[x(n)]$. As Plotkin and Swamy proposed a more general class for $O[x(n)]$, with $|l - q| = |p - s| = b$ for any integer $b \neq 0$ [7], we refer to the operator $\Theta[x(n)]$ as the Agarwal–Gotman operator [6].

A point worth considering. Agarwal and Gotman stated that $\Theta[A\cos(\omega_0 n) + \phi] = A^2\sin^2(\omega_0)$ [6]; we find that $\Theta[A\cos(\omega_0 n) + \phi] = 2A^2\sin^2(\omega_0)\cos(\omega_0)$. And $\Theta[A\cos(\omega_0 n) + \phi] \approx A^2\omega_0^2$ for $0 \leq \omega\pi/6.33$. We arrived at this range experimentally to yield an approximate error of 11%. The Agarwal–Gotman operator therefore requires up-sampling by a factor of 6.33, greater than the factor of 4 for the Teager–Kaiser operator.

The up-sampling process transforms white to coloured noise, and in this case

$$\mathbb{E}\{\Psi[y(n)]\} = \Psi[y(n)] + \sigma^2[R_{ww}(0) - R_{ww}(2)]$$
$$\mathbb{E}\{\Theta[y(n)]\} = \Theta[y(n)] + \sigma^2[R_{ww}(1) - R_{ww}(3)]$$

where $R_{ww}(n)$ is the autocorrelation function of $w(n)$. If we incorporate the necessary up-sampling, then $T_1 = 6.33T_2/4$ where T_1 is the sampling period for the Teager–Kaiser operator and T_2 is the sampling period for the Agarwal–Gotman operator, then $\mathbb{E}\{\Psi[w(n); T_1]\} = \sigma^2[R_{ww}(0) - R_{ww}(2T_1)]$ and $\mathbb{E}\{\Theta[w(n); T_2]\} = \sigma^2[R_{ww}(T_2) - R_{ww}(3T_2)] = \sigma^2[R_{ww}(4T_1/6.33) - R_{ww}(12T_1/6.33)]$. Assuming that $R_{ww}(m)$, for $m = 0, 1, 2, 3$, is monotonically decreasing, then $\sigma^2[R_{ww}(0) - R_{ww}(2T_1)] > \sigma^2[R_{ww}(T_2) - R_{ww}(3T_2)]$. Both operators will have a bias term although this bias term will be smaller for the Agarwal–Gotman operator.
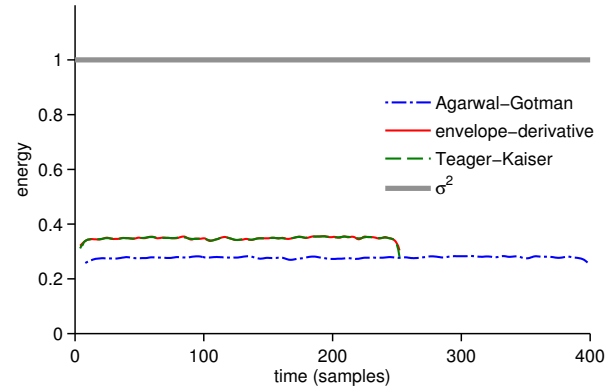

Fig. 2. Expectation of the three energy operators for white Gaussian noise. Noise was zero mean with a variance (σ^2) of one. The expectation operation was approximated by averaging over 10 000 realisations. Each noise signal, of length 64 samples, was up-sampled prior to analysis thus converting the white noise to coloured noise. Noise was up-sampled by a factor of 4 for the Teager–Kaiser and envelope–derivative operators and by a factor of 6.33 for the Agarwal–Gotman operator.
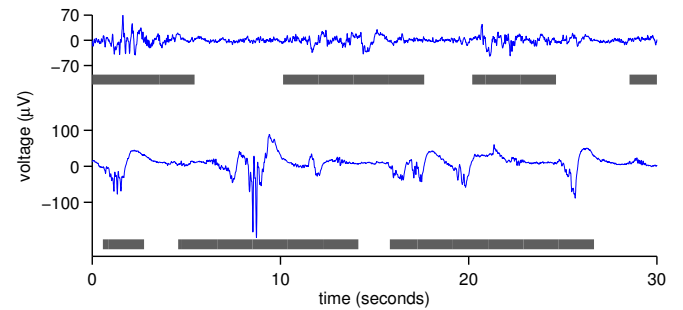

Fig. 3. Examples of EEG with expert annotations (grey blocks). Top: EEG from a term baby showing tracé alternant with annotations below the signal highlighting the high-voltage activity. Bottom: EEG from a preterm baby with annotations highlighting burst activity.

Simulations in Fig. 2 highlight similar conclusions and show that the Teager–Kaiser and envelope–derivative operators have similar bias-terms.

III. DETECTION APPLICATION FOR NEWBORN EEG

An application of energy measures is the detection of high-voltage activity in the EEG of newborn infants [2], [8]. For this application, we selected a data set from existing EEG recorded at the neonatal intensive care unit in Cork University Maternity Hospital (CUMH), Ireland. Informed parental consent was obtained prior to EEG data collection and ethical approval was obtained from CUMH and University College Cork. All data was anonymised at time of recording.

The data set consisted of 2-minute epochs from 10 preterm (premature) infants and 10 full-term infants. Expert electroencephalograghers annotated burst activity in the preterm data and high-voltage activity in the tracé alternant pattern of the full-term data. One channel from each EEG record was used in the analysis. EEG was
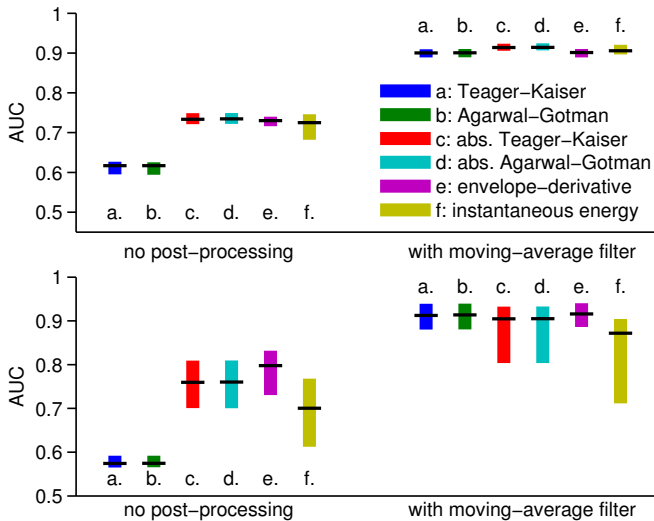
Fig. 4. Detection performance of different methods. Results for detecting high-voltage activity in the tracé alternant pattern of fullterm EEG data (top plot) and burst-activity in preterm EEG data (bottom plot). Each data set includes 2 minute EEG recordings from 10 babies. The area-under the (receiver-operating characteristic) curve (AUC) measures detection performance; the coloured, thick vertical lines represent the inter-quartile range and the black, thin horizontal lines represent the median. Post-processing options include taking the absolute (abs.) value of the operator and applying a 1.5 second moving-average filter.

recorded with a sampling frequency of 256 Hz and was low-pass filtered at 20 Hz, using a 6-order elliptic filter, to avoid high-frequency artefacts.

We compared the proposed envelope–derivative operator to the existing Teager–Kaiser operator and its variants. Specifically, we assessed the detection performance of the Teager–Kaiser operator in (4) and the Agarwal–Gotman operator in (5). And for comparison, we included an instantaneous energy measure (square of the signal) without frequency weighting [8]. In addition, we applied two different post-processing methods: taking the absolute value of the operators (not applicable to the non-negative envelope–derivative operator) and by applying a low-pass filter, in the form of 1.5 second moving-average window. These post-processing steps were proposed for detecting bursts in preterm EEG using the Agarwal–Gotman operator [2].

IV. RESULTS AND DISCUSSION

Fig. 4 shows the results for both data sets. Performance was assessed on time-based agreement (sample-by-sample) using the area-under the receiver-operator characteristic curve (AUC). Our first observation is that the Teager–Kaiser and Agarwal–Gotman operators have almost identical performance across all tests. We expect this result in a detection application when noise is present throughout and, or, when both operators have approximately similar noise bias. Second, performance for both the Teager–Kaiser and Agarwal–Gotman operators, in their original form, is poor in comparison to the envelope–derivative operator. This may be caused by the

fact that the Teager–Kaiser and Agarwal–Gotman operators can be negative and are more oscillatory. Applying the absolute value or a low-pass filter helps improve performance dramatically, increasing performance on the preterm data, for example, from 0.57 to 0.76 AUC.

Third, for the full-term data, performance of the instantaneous energy measure is similar to the frequency-weighted energy measures. Thus, there may be little difference in frequency between the low- and high-voltage periods in the tracé alternant pattern. And the final observation is that all frequency-weighted measures have similar performance after post-processing.

In conclusion, the non-negative property of the envelope–derivative has proved to be advantageous: the absolute value of Teager–Kaiser operator is needed to reach similar performance to the envelope–derivative operator. One disadvantage of the envelope–derivative operator is that the discrete definition requires a long-duration Hilbert transform filter, thus increasing latency for real-time implementations. Design of an effective short-duration Hilbert transform filter for the operator could be explored if required. But for many—if not all—EEG applications, real-time implementation is not a priority and the envelope–derivative operator presents an ideal measure to assess instantaneous, frequency-weighted energy. Computer code for all methods is available at http://otoolej.github.io/code/nleo/.

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